The direct concern of our chapter is not with unrestricted quantification. Or at least: not *unrestrictedly unrestricted* quantification—quantification over absolutely everything there is. Our direct concern is with whether it is admissible to quantify over *sets* without restriction—whether it is coherent to speak of, and have bound variables ranging over, all pure sets, or all pure set-like totalities (see also Shapiro, 2003). (‘Well, but didn’t you just do so?’) Closely connected questions, also within our focus, are whether it makes sense—and if so, what kind of sense—to speak of all *cardinal numbers*, or all cardinalities, and all *ordinal numbers*, or all order-types of well-orderings?

That such quantification is somehow illicit has of course often been suggested as the principal lesson to be taken from the Russell, Cantor, and Burali-Forti paradoxes. And if quantification over all sets, for example, is indeed illicit, so much the worse, presumably, for the even more ambitious ‘absolutely everything’. (Of course even if quantification over all sets is permissible, there could still be residual problems for ‘absolutely everything’. But we will not attempt to explore the territory opened by that observation here.)

### 10.1 INDEFINITE EXTENSIBILITY INTUITIVELY UNDERSTOOD

In a much-discussed letter to Dedekind, Cantor wrote:

... it is necessary, as I discovered, to distinguish two kinds of multiplicities ... For a multiplicity can be such that the assumption that *all* of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities*. As we can readily see, the 'totality of everything thinkable', for example, is such a multiplicity ...

If on the other hand the totality of elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into 'one thing', I call it a *consistent multiplicity* or a 'set'.

(Cantor, 1899, 114)
This connotation of the word ‘set’ is now standard, and we will stick to it here. Cantor’s distinction between sets and ‘inconsistent multiplicities’ goes back at least to his (1883). Prior to that, he only considered sets of some fixed ‘conceptual sphere’, such as sets of natural numbers or sets of real numbers (see Tait, 2000). In the 1883 Grundlagen, he wrote that ‘By a ‘manifold’ or ‘set’ I understand any multiplicity which can be thought of as one, i.e., any aggregate of determinate elements which can be unified into a whole by some law’. In defining the transfinite numbers, Cantor invoked two principles. The first is that each number α has an immediate successor α + 1. The second is that each set S of numbers which has no largest member has a limit: the smallest number larger than every member of S. It follows immediately that the transfinite numbers cannot compose a set and are thus an inconsistent multitude. As Cantor somewhat obliquely expresses it, the numbers are the result of ‘a thoroughly endless process of creation’. His work presupposes that every number is ‘generated’ from one of these two principles.

A few years later, Russell provided a more nuanced characterization of what appears to be essentially the same idea. His (1906) begins with an examination of the standard paradoxes, and concludes:

the contradictions result from the fact that . . . there are what we may call self-reproductive processes and classes. That is, there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect all of the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also having the said property.

Citing this passage, Michael Dummett (1993, 441), writes that an indefinitely extensible concept is one such that, if we can form a definite conception of a totality all of whose members fall under the concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it

(emphasis ours).

According to Dummett, an indefinitely extensible concept P has a ‘principle of extension’ that takes any definite totality t of objects each of which has P, and produces an object that also has P, but is not in t (see also Dummett, 1991, 316–19). Let us say that a concept P is Definite if it is not indefinitely extensible. Connecting this with Cantor’s terminology, we can say that P is Definite if and only if the P’s are ‘consistent’, and thus form a set.

Obviously, Dummett’s remarks won’t do as a definition, since he uses the complementary ‘definite’ to characterize what it is for a concept to be indefinitely extensible. And Russell, of course, does no better by speaking unqualifiedly of ‘any class of terms all having such a property’, since he means us to take it as given that classes, properly so regarded, are ‘wholes’—are Definite. But the following familiar material makes salient the pattern that Russell and Dummett both discern:

(1) The Burali-Forti paradox. Rather than work with the usual identification of the ordinals with sets in the iterative hierarchy, such as von Neumann ordinals, let us here think of them in an intuitive way simply as order-types of well-orderings: an ordinal
All Things Indefinitely Extensible

is an object denoted by a nominalization of a predicate for a well-ordering. Let $O$ be any definite collection of ordinal numbers. Let $O'$ be the collection of all ordinals $\alpha$ such that there is a $\beta \in O$ for which $\alpha \leq \beta$. It is easy to see that $O'$ is well-ordered under the natural ordering of ordinals. Let $\gamma$ be the order-type of $O'$. So $\gamma$ is itself an ordinal. Let $\gamma'$ be the order-type of $O' \cup \{\gamma\}$. That is $\gamma'$ is the order-type of the well-ordering obtained from $O'$ by tacking an element on at the end. Then $\gamma'$ is an ordinal number, and $\gamma'$ is not a member of $O$. So ordinal number is indefinitely extensible.

As Dummett (1991, 316) puts it,

if we have a clear grasp of any totality of ordinals, we thereby have a conception of what is intuitively an ordinal number greater than any member of that totality. Any [D]efinite totality of ordinals must therefore be so circumscribed as to forswear comprehensiveness, renouncing any claim to cover all that we might intuitively recognise as being an ordinal.

In the next section we will offer an argument that the notion of ordinal number is in fact the central paradigm of an indefinitely extensible concept.

(2) The Russell Paradox. Let $R$ be any set of sets that do not contain themselves; so if $r \in R$ then $r \not\in r$. Then $R$ does not contain itself. So the concept, set that does not contain itself, is indefinitely extensible—any set of such sets omits a set, namely itself. A fortiori, set itself is indefinitely extensible, since any definite collection—set—of sets must omit the set of all of its members that do not contain themselves.

(3) The Cantor Paradox. Let $C$ be a collection of cardinal numbers. Let $C'$ be the union of the result of replacing each $\kappa \in C$ with a set of size $\kappa$. The collection of subsets of $C'$ is larger than any cardinal in $C$. So cardinal number is indefinitely extensible.

To be sure—though perhaps with the exception of the reasoning leading to Russell’s paradox—these examples are not completely uncontentious. One can challenge the set-theoretic principles (Union, Replacement, Power-set, etc.) that are invoked in the constructions. Or else one can tinker with the logic. Nevertheless, we think it reasonable to agree with Russell and Dummett that the concepts in question do have the ‘self-reproductive’ feature which the notion of indefinitely extensibility gestures at.

There remain the questions

(i) Whether the notion can be characterized more satisfactorily, without circularity;

(ii) Which are the indefinitely extensible concepts/totalities when the notion is best understood,

and

(iii) What bearing the notion has on the various issues in the philosophy of mathematics, including

- the proper diagnosis of the paradoxes;
- the legitimacy of unrestricted quantification;
- the content of quantification (if legitimate at all) over indefinitely extensible totalities and the legitimacy of classical logic for such quantifiers;
- the proper conception of the infinite; and
- the possibilities for neo-logicist foundations for set-theory,
on each of which it has been held to have some import. A satisfactory treatment of this agenda would be a task for a substantial book. But each of the listed issues will be touched on, if only modestly, in the discussion to follow.

10.2 INDEFINITE EXTENSIBILITY AND THE ORDINALS: RUSSELL’S CONJECTURE

Russell (1906, 144) wrote that it ‘is probable’ that if $P$ is any concept which demonstrably does not have an extension, then ‘we can actually construct a series, ordinally similar to the series of all ordinals, composed entirely of terms having the concept $P$’. Putting the presumably metaphorical talk of construction aside,¹ Russell’s conjecture is in effect that if $P$ is indefinitely extensible, then there is a one-to-one function from the ordinals into $P$. Russell does not give an argument for this, but here is one:

Let $\alpha$ be an ordinal and assume that we have a one-to-one function $f$ from the ordinals smaller than $\alpha$ to objects that fall under $P$. Consider the collection $\{f\beta \mid \beta < \alpha\}$. This is Definite. Since $P$ is indefinitely extensible, there is an object $a$ such that $P$ holds of $a$, but $a$ is not in this set. Set $f\alpha = a$.

Thus for any ordinal $\alpha$, if all the ordinals smaller than $\alpha$ can be injected into $P$, then the ordinals up to and including $\alpha$ can be injected into $P$. So all the ordinals can be injected into $P$.

This reasoning requires transfinite recursion on ordinals—but that can hardly be doubted. It is part of what it is to be an ordinal that definitions by transfinite recursion and proofs by transfinite induction are valid. (Both points seem to follow from Cantor’s two principles noted above together with the assumption that every ordinal is ‘constructed’ from one or the other of them.) The argument also relies on a version of Replacement: if a totality $t$ is equinumerous with an ordinal, then $t$ is Definite.²

¹ This and cognate metaphors are pervasive in the source literature. Cantor speaks of a process of ‘creation’ of ever more numbers. Russell notes that indefinitely extensible concepts come with ‘processes’ which ‘seem essentially incapable of terminating’, and Dummett speaks of ‘principles of extension’. The metaphors are stretched. It is stretching things, for example, to think of the ordinals, classically conceived, as generated by a process. Processes take place in time, and time does not have enough structure to carry the ‘construction’ of ordinals very far into the transfinite (see Parsons, 1977). A major part of the interpretative and analytical project in the vicinity is to provide a satisfactory reading of the intent of these metaphors.

² Depending on the exact formulation of the notion of indefinite extensibility, the argument might also invoke a global choice principle, or at least a choice function on sub-totalities of the given indefinitely extensible property. Recall the clause from the first passage from Russell: ‘there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question’. If this means something like ‘given any class of terms all having such a property, there is a new term also having the property in question’, Choice is needed in the argument. On the other hand, if Russell’s clause is taken (more or less) literally, the ability to ‘define’ a new term is part of what it is for a notion to be indefinitely extensible. If so, then Choice is not needed in the argument. Similarly, Dummett says that an indefinitely extensible property $P$ has a ‘principle of extension’ that takes any Definite totality $t$ of objects each of which has $P$, and produces an object that also has $P$, but is not in $t$. If this principle of extension is a function, then
There is no knowing whether Russell had something like the above argument supporting his conjecture in mind. Whether or not he did, later in the same article, he invokes the conjecture to motivate a ‘limitation of size’ resolution of the paradoxes. But, as we saw, the above argument invokes Replacement, which itself expresses a limitation of size principle. So for Russell’s own purpose, the argument offered him above would beg the question.³

The converse of Russell’s conjecture seems solid for its part: if there is a one-one function taking the ordinals into the objects satisfying a concept $P$, then $P$ is indefinitely extensible. For let $f$ be a one-to-one function from the ordinals into the $P$’s, and let $C$ be a Definite collection of $P$’s. Let $c$ be the collection of ordinals $\alpha$ such that $f\alpha$ is in $C$. By Replacement, $c$ is a Definite collection (and thus a set) of ordinals. Let $\alpha'$ be the smallest ordinal not in $c$. Then $Pf\alpha'$, but $f\alpha'$ is not in $C$. If the ordinals can be embedded into the $P$’s, then $P$ inherits the indefinite extensibility of the ordinals.

So both Russell’s conjecture and its converse are plausible. Together they imply that a concept is indefinitely extensible if and only if there is an injection of the ordinals into it. This is the reason, foretold in the previous section, to take the ordinals to be the paradigm case of an indefinitely extensible totality, and the mechanics of the Burali–Forti paradox to be the paradigm of indefinite extension.

Historical note: Cantor himself made frequent use of the converse of the Russell conjecture. To show that a given multiplicity is ‘inconsistent’, he would show how to embed the series of ordinals into it. The aforementioned letter to Dedekind contains a sketch of the series of alephs. Cantor asks whether there is a set whose cardinality is not an aleph:

This question is to be answered negatively . . . If we take a definite multiplicity [i.e., a set] $V$ and assume that no aleph corresponds to it as its cardinal number, we conclude that $V$ must be inconsistent. For we readily see that, on the assumption made, the whole system $\Omega$ [of transfinite numbers] is projectible into the multiplicity $V$, that is, there must exist a submultiplicity $V'$ of $V$ that is equivalent to the system $\Omega$.

Cantor thus provides an argument for what would later be Zermelo’s (1904) well-ordering theorem. In his capacity as editor of Cantor’s collected works, Zermelo added a long footnote to the published version of Cantor’s letter taking him to task for his talk of ‘procedures’ and the like:⁴

Cantor apparently thinks that successive and arbitrary elements of $V$ are assigned to members of $\Omega$ in such a way that every element of $V$ is used only once. Either this procedure would come we do not need Choice in the foregoing argument. The presumed principle of extension does the ‘choosing’.

³ The ‘limitation of size’ conception is that a given ‘totality’ forms a set if and only if it is not too ‘large’. From this perspective, the sets, the ordinals, and the cardinals do not themselves form sets because there are too many of them. Russell’s conjecture fits in well with this conception. If it were the case that the ordinals could be embedded into any indefinitely extensible totality $T$, then, since there are ‘too many’ ordinals to form a set, there are also too many $T$’s to form a set. The replacement principle is a contrapositive, of sorts, to this observation. It says that if a collection $S$ is Definite, and there are exactly as many $U$’s as $S$’s, then the $U$’s form a set as well. Thanks to Michael Potter here.

⁴ Cantor’s letter and Zermelo’s note are published in van Heijenoort (1967, 113–17).
to an end once all elements of $V$ had been exhausted, and then $V$ would be mapped onto a segment of the number sequence and its cardinality would be an aleph, contrary to the assumption, or $V$ would remain inexhaustible, hence contain a constituent part that is equivalent to all of $\Omega$ and therefore inconsistent. Thus the intuition of time is applied here to a process that goes beyond all intuition . . . Only through the ‘axiom of choice’, which postulates the possibility of a simultaneous choice and which Cantor uses unconsciously and instinctively everywhere but does not formulate explicitly anywhere, could $V'$ be defined as a subset of $V$. But even then there would still remain a doubt: perhaps the proof involves ‘inconsistent’ multiplicities, indeed possibly contradictory notions, and is logically inadmissible already because of that. It is precisely doubts of this kind that impelled the editor (Zermelo) a few years later to base his own proof of the well-ordering theorem (1904) purely upon the axiom of choice without using inconsistent multiplicities.

10.3 ‘SMALL’ INDEFINITELY EXTENSIBLE CONCEPTS? CASE STUDY (1): THE BERRY PARADOX

Russell’s Conjecture, taken with its converse, makes for an extensional connection between ordinal and indefinite extensibility: the totality of elements falling under an indefinitely extensible concept contains a system isomorphic to the ordinals. Clearly, though, even if this connection is accepted, we still want for an analysis, or further conceptual elucidation, of the notion: something to provide some leverage on the cluster of issues itemized at the conclusion of Section 10.1—something the connection made with the ordinals, unsupplemented, manifestly fails to do.

But is the connection made by Russell’s Conjecture correct in any case? Dummett for one has characteristically taken it that the natural and real numbers are indefinitely extensible totalities in just the same sense that the ordinals and cardinals are, with similar consequences, in his view, for the understanding of quantification over them and the standing of classical logic in the investigation of these domains. And in the article (Dummett, 1963) which contains his earliest published discussion of the notion, he argues that the proper interpretation of Gödel’s incompleteness theorems for arithmetic is precisely to teach that arithmetical truth and arithmetical proof are indefinitely extensible concepts—yet neither presumably has an even more than countably infinite extension, still less an ordinals-sized one. (For the ordinary, finitely based language of second-order arithmetic presumably suffices for the expression of any arithmetical truth.) It is disconcerting to have lost contact with one of the leading friends of indefinite extensibility so early in the discussion. But then where does the argument for Russell’s conjecture and its converse go wrong, or betray Dummett’s intent?

It is relevant to recall that Russell (1908) himself, in motivating a uniform diagnosis of the paradoxes, included in his list of chosen examples some at least where the ‘self-reproductive’ process seems bounded by a relatively small cardinal. For instance the Richard paradox concerning the class of decimals that can be defined by means of a finite number of words makes play with a totality which, if indeed indefinitely extensible, is at least no greater than the class of decimals itself, i.e. than $2^{\aleph_0}$. Was
Russell simply unaware of this type of example in 1906, when he proposed the Conjecture discussed above? Or did he not in 1906 regard the Richard paradox and others involving ‘small’ totalities as genuine examples of the same genre, revising that opinion two years later?

Well, are they examples of the same genre? To fix ideas, let us consider in some detail the so-called Berry paradox, the paradox of ‘the smallest natural number not denoted by any expression of English of fewer than seventeen words’. This is an English expression which, on plausible assumptions, should denote a natural number—but it contains sixteen words. So its referent—the smallest natural number not denoted by any expression of English of fewer than seventeen words—is denoted by an English expression of sixteen words. Is this a paradox of indefinite extensibility?

Let’s try to state the paradox more carefully. Define an expression \( t \) to be numerically determinate if \( t \) denotes a natural number and let \( C \) be the set—if there is one—of all numerically determinate expressions of English. Consider the expression \( b \): ‘the smallest natural number not denoted by any expression in \( C \) of fewer than seventeen words’. Assume: (1) that \( b \) is a numerically determinate expression of English (i.e., \( b \in C \)) and (2) that \( C \) indeed exists. Then contradiction follows from (1) and (2) and the empirical datum that \( b \) has sixteen words (counting the contained occurrence of ‘\( C \)’ as one word).

Evidently, there are some issues that would need to be addressed in a watertight version of the paradox. Notice, for instance, that assumption (1) presupposes that ‘\( C \)’ is an expression of English. But, of course, you won’t find it in any dictionary of the English language. The paradox presumes to take ‘English’ to include expressions introduced by new explicit abbreviative definitions of English expressions. Assumption (1) also presupposes of course that \( b \)—‘the smallest natural number not denoted by any expression in \( C \) of fewer than seventeen words’—denotes something in English. But what in that case is being assumed about the decimal numerals? Are they part of English? If so, how many words do they contain? If e.g. ‘1002’ ranks as a one-word English expression, then assumption (1) is contradicted and the paradox is stillborn.

We could take the view that each occurrence of one of the ten basic decimal numerals within a compound decimal numeral counts as one word. But once we allow English to contain abbreviative definitions, what is to stop us taking an infinite series of—presumably—single-symbol expressions,

\[
\begin{array}{cccccc}
T & T & T & T & T & T
\end{array}
\]

asymptotically approaching a size half that of the first, and then assigning to each \( n^{th} \) expression in this series the \( n^{th} \) natural number as referent and deeming the whole to be part of English? In that case, \( b \) has again no denotation in English.

But suppose we manage to work these issues out and get a rigorous contradiction. Then the diagnostic thought, shared by Russell and Dummett, will be that the paradox is due to the indefinite extensibility of the concept, numerically determinate expression of English. The problem, on this proposal, is the assumption that any such set as \( C \)—the set of all numerically determinate expressions of English—exists. There is
no such set. Rather, any set of such expressions aids and abets the construction of a new such expression, in the kind of way illustrated by the paradox, which it demonstrably cannot contain.

The analogy with the classic paradoxes looks good. But, as emerges if we think the process of ‘indefinite extension’ through, it is not quite right. Again: take $P$ as the concept, numerically determinate expression of English, and let $D$ be any definite, finite collection of $P$’s. Introduce a (one word) name $d$ for $D$ (counting $d$, as remarked above, as part of ‘English’), and consider ‘the smallest natural number not denoted by any member of $d$ of fewer than seventeen words’. Call this sixteen-worded expression ‘$w$’. It is clear that $w$ is a $P$ (i.e., a numerically determinate expression of English); for there will be a definite finite subset, $D^*$, of $D$ comprising exactly those of the members of $D$ with fewer than seventeen words, and—since all these expressions are numerically determinate—a definite greatest number, $k$, denoted by any member of $D^*$. So the referent of $w$ is less than or equal to $k + 1$. However $w$ is not in $D$, since no member of $D$ denoting $w$ has fewer than seventeen words. Clearly we can iterate the process, now taking $D \cup \{w\}$ for $D$, and giving this set a new one-word English name. Still the process is not indefinitely extensible.

To see why, let the initial collection $D$ just consist of the Arabic numerals, ‘0’–‘9’. Do the Berry construction on this to get a $w_1$ — it will denote 10 — that is $P$, but not in $D$. Let $D_1$ be $D \cup \{w_1\}$. Give $D_1$ a one-word name. Now do a Berry on $D_1$, producing $w_2$. Let $D_2$ be $D_1 \cup \{w_2\}$. Give $D_2$ a one-word name. Do the Berry construction again. Keep going ... Now let $D_\omega$ be the union of $D, D_1, D_2, \ldots$ What happens next? — what happens when we apply the Berry construction to $D_\omega$?

The answer is that it fails. For reflect that 0 to 9 are all denoted by single-word members of $D$; 10 is denoted by the sixteen-worded ‘the smallest natural number not denoted by any member of [write in the one-word name of $D$] of fewer than seventeen words’; 11 is denoted by the ‘the smallest natural number not denoted by any member of [write in the one-word name of $D_1$] of fewer than seventeen words’; 12 is denoted by the ‘the smallest natural number not denoted by any member of [write in the one-word name of $D_2$] of fewer than seventeen words’; and so on. So the ‘the smallest natural number not denoted by any member of [write in the one-word name of $D_\omega$] of fewer than seventeen words’ has no reference — for every natural number is denoted by at least one member of $D_\omega$ of fewer than seventeen words.

Of course the construction fails in a more pedestrian way if we do not allow ourselves to include as part of English the countably many one-word abbreviations needed to press the iteration beyond the finite. What it seems fair to say is that, with that idealization — if that is what it is — of what counts as English, there is a kind of indefinite extensibility about the concept, numerically determinate expression of English. But it is a bounded indefinite extensibility, as it were — indefinite extensibility up to a limit (ordinal). If union is a Definiteness preserving operation, there will be, in such bounded cases, a definite collection of entities of the kind in question that does not in turn admit of extension by the original operation. So they will not be indefinitely extensible, at least not in the spirit of our initial characterization.
10.4 ‘SMALL’ INDEFINITELY EXTENSIBLE CONCEPTS?

CASE STUDY (2): ARITHMETICAL TRUTH

As noted earlier, Dummett (1963) argues that Gödel’s incompleteness theorem shows that arithmetical truth is indefinitely extensible. Given any definite collection C of arithmetical truths, one can construct a truth—the Gödel sentence for C—that is not a member of C.

This is prima facie a puzzling claim. If ‘definite collection’ means something like set, and if the latter concept is understood as in classical mathematics, then it just seems wrong—arithmetical truth is not indefinitely extensible. Following Tarski, one can give a straightforward explicit definition of ‘arithmetical truth’. It then follows from the Aussonderungssaxiom that there is a set of all arithmetical truths. There is no ‘Gödel sentence’ for this set. But, of course, Dummett’s claim is not offered, presumably, within the context of the classical conception of set.

Fix an effective Gödel numbering of the sentences of arithmetic. It follows from Tarski’s theorem that the notion of arithmetical truth is not arithmetic.⁵ In other words, there is no formula $T(x)$ in the language of arithmetic, such that for each natural number $n$, $T(n)$ if and only if $n$ is the Gödel number of a truth of arithmetic.

A generalization of Gödel’s theorem does suggest something approximating indefinite extensibility: if $C$ is any arithmetic set of (Gödel numbers of) truths of arithmetic, then there is a truth of arithmetic that is not in $C$. Indeed, let $A(x)$ be a formula in the language of arithmetic that characterizes $C$. That is, for every natural number $n$, $A(n)$ if and only if $n$ is the Gödel number of a true sentence of arithmetic. Let $P$ be a fixed point for the formula $\neg A(x)$. That is, if $p$ is the Gödel number of $P$, then

$$P \equiv \neg A(p)$$

is a theorem of arithmetic. It follows that $P$ is true,⁶ and thus $\neg A(p)$. So $p$ is not in $C$.

Given Dummett’s general outlook in the philosophy of mathematics, he would surely deny that the ‘totality’ of arithmetical truths is Definite. It is certainly not decidable. Perhaps it is reasonable to hold that, for Dummett, a ‘totality’ of natural numbers is Definite only if it is recursively enumerable, or at least arithmetic. If so, the foregoing construction shows that, for Dummett, something in the neighborhood of indefinite extensibility holds for arithmetical truth.

⁵ A property (or set) of natural numbers $F$ is arithmetic if there is a formula $\Phi(x)$ in the language of arithmetic, with only $x$ free, such that for each natural number $n$, $n$ is an $F$ (or $n \in F$) if and only if $\Phi(n)$ holds. The general form of Tarski’s (1933) theorem is that no sufficiently rich interpreted language can define its own truth predicate. The specific form here is that there is no arithmetic definition of ‘truth in arithmetic’.

⁶ Assume $A(p)$. Then $p$ is the code of a truth of arithmetic. But $p$ is the code of $P$. So $P$ is true. But $P$ is equivalent to $\neg A(p)$, which is thus true. Contradiction. So $\neg A(p)$, which (again) is equivalent to $P$. 
More specifically, it is straightforward to initiate something that looks like a process of ‘indefinite extension’. Let \( A_0 \) be a given Definite set of arithmetical truths—for instance, let \( A_0 \) be the theorems of some standard axiomatization of arithmetic. For each natural number \( n \), let \( A_{n+1} \) be the collection \( A_n \) together with a Gödel sentence for \( A_n \). Presumably, if \( A_n \) is Definite, then so is \( A_{n+1} \), and, of course, \( A_n \) and \( A_{n+1} \) are distinct. Unlike the above situation with the Berry paradox, this ‘construction’ can be continued into the transfinite. Let \( A_\omega \) be the union of \( A_0, A_1, \ldots \) Arguably, \( A_\omega \) is Definite. Indeed, if \( A_0 \) is recursively enumerable, then so is \( A_\omega \); if \( A_0 \) is arithmetic, then so is \( A_\omega \). Thus, we can define \( A_{\omega+1}, A_{\omega+2}, \ldots \) Then we take the union of those to get \( A_{2\omega} \), and onward, Gödelising all the way (so to speak).

On the usual, classical construal of the extent of the ordinals, however, the ‘construction’ does not continue without limit. It ‘runs out’ well before the first uncountable ordinal. Let \( \lambda \) be an ordinal and let us assume that we have defined \( A_\lambda \). The foregoing construction will take us on to the next set \( A_{\lambda+1} \) only if the collection \( A_\lambda \) has a Gödel sentence. And this is possible only if \( A_\lambda \) is arithmetic. Clearly, it cannot be the case that for every (countable) ordinal \( \lambda \), \( A_\lambda \) is arithmetic. There are only countably many arithmetic sets (at most one for each formula in the language of arithmetic), but there are uncountably many (countable) ordinals.⁷ Let \( \kappa \) be an ordinal such that \( A_\kappa \) is not arithmetic. For a Dummettian, presumably, \( A_\kappa \) is not Definite.

One option, to be sure, is that of accepting the Russell-conjecture but maintaining that there is no such ordinal as \( \kappa \). The proof, in set theory, that such an ordinal exists relies on excluded middle. But, for the classical mathematician at least, the notion of arithmetical truth is not fully indefinitely extensible; we cannot run on indefinitely through the ordinals in iterating Gödel sentences.

10.5 INDEFINITE EXTENSIBILITY EXPLAINED

Let’s take stock. Russell’s Conjecture, that indefinitely extensible concepts are marked by the possession of extensions into which the ordinals are injectible, still stands. Apparent exceptions to it, like numerically determinate expression of English and arithmetical truth, are not really exceptions. For the principles of extension they involve are not truly indefinitely extensible but stabilize after some series of iterations isomorphic to a proper initial segment of the ordinals. Or at least they do so if the ordinals are allowed their full classical structure. As we just noted, the friend of small indefinitely extensible concepts has the option of preserving Russell’s Conjecture by

---

⁷ We can be a bit sharper. If \( \lambda \) is a limit ordinal, then the contents of \( A_\lambda \), if it exists, depend on the particular Gödel numbering chosen and, more importantly, on the method for coding countable ordinals as natural numbers (so we can ‘axiomatize’ \( A_\lambda \)). For any such coding, there are countable ordinals that have no code. Let \( \kappa \) be the smallest such (for a given coding). Then for each \( \lambda < \kappa, A_\lambda \) exists and is arithmetic. So we can go on to \( A_{\lambda+1} \). We can take the ‘union’ of all such sets, which is \( A_\kappa \). But this is the end of the line: \( A_\kappa \) is not arithmetic, and thus has no Gödel sentence. The construction sketched here, of iterating the Gödel construction into the transfinite, is well-studied. For more details, see Turing (1939) and Feferman (1962), (1988).
‘cutting back’ the ordinals appropriately far. At the limit, when only the finite ordinals are countenanced, there will then be many more indefinitely extensible concepts than otherwise, including *numerically determinate expression of English* and *arithmetical truth*; and all will, indeed, be ‘small’. Any invocation of the notion of ordinal number in the *explanation* of indefinite extensibility may seem to invert the priorities, but actually there is something importantly right about it. Intuitively the indefinite extensibility of $P$ has to do with the $P$-conservativeness of some germane principle of extension *no matter how long a series* of iterated applications of it may be made. What one thinks that means will inevitably depend on how one thinks about the structure of the *measures* of such a series of iterated applications—and so will depend on one’s preconceptions about ordinal number. More or less generous such preconceptions will consequently factor into the extension of indefinite extensibility. This relativity, we suggest, was inbuilt from the start, and the concept of indefinite extensibility is consequently open to refinement and mutation in tandem with developments, sophisticated or unfortunate, in one’s conception of how long such series can in principle be (see Section 10.9 below).

That said, though, Russell’s Conjecture, even if extensionally correct, is not the kind of characterization of indefinite extensibility we should like to have. To get a clear sense of the shortfall, reflect that if Russell’s Conjecture provided a full account, it would be a *triviality* that the ordinals are indefinitely extensible. Whereas what is wanted is a perspective from which we can explain *why* Russell’s Conjecture is good if indeed, as it seems, it is—equivalently, a perspective from which we can explain what it is about *ordinal* that *makes it* the paradigm of an indefinitely extensible concept.

Any indefinitely extensible totality $P$ is intuitively unstable, ‘restless’, or in ‘growth’. Whenever you think you have it safely corralled in some well-fenced enclosure, suddenly—hey presto!—another fully $P$-qualified candidate pops up outside the fence. The primary problem in clarifying this figure is to dispense with the metaphors of ‘well-fenced enclosure’ and ‘growth’. Obviously a claim is intended about sub-totalities of $P$ and functions on them to (new) members of $P$. Equally obviously, we need to qualify for which type of sub-totalities of $P$ the claim of iterative extensibility within $P$ is being made. Clearly it cannot be sustained for absolutely any sub-totality of $P$: if for example, we continue to take it that *ordinal* is a paradigm of indefinite extensibility, we do not claim that *ordinal* itself picks out a sub-totality of the relevant kind (though of course there are issues, which will occupy us later, about whether one can avoid that claim). Nor would it help to restrict attention to proper sub-totalities: *ordinal* other than three does not pick out the right kind of sub-totality either. If we could take it for granted that the notion of indefinite extensibility is in clear standing and picks out a distinctive type of totality, or concept, then we could characterize the relevant kind of sub-totality exactly as Dummett did—they are the *Definite* sub-totalities. For the indefinite extensibility of a totality, if it consists in anything, precisely consists in the fact that any Definite sub-totality is merely ‘proper’. But unless there is some direct route into the intended notion of Definiteness other than via ‘not indefinitely extensible’ we make no explanatory progress. No doubt circularity in our best explanation of a concept need not—pace Quine (1951)—enforce skepticism about it. Concepts can be explained by giving illustrative instances, for one thing. But the problem in
this case is that the intended concept is too sophisticated to allow of explanation only by examples: what the standard examples seem to illustrate—if indeed they genuinely illustrate anything distinctive at all—cannot be a basic resemblance, beyond further articulation, but surely has to be something which allows of explicit characterization.

What is the way forward? Here is our suggestion. In order, at least temporarily, to finesse the ‘which sub-totalities?’ issue, let’s start with an explicitly relativized notion. Let \( P \) be a concept of items of a certain type \( \tau \). Typically, \( \tau \) will be the (or a) type of individual objects. Let \( \Pi \) be a concept of concepts of type \( \tau \) items. Let us say that \( P \) is indefinitely extensible with respect to \( \Pi \) if and only if there is a function \( F \) from items of the same type as \( P \) to items of type \( \tau \) such that if \( X \) is any sub-concept of \( P \) such that \( /\Pi X \), then

\[
\begin{align*}
(1) & \quad FX \text{ falls under the concept } P, \\
(2) & \quad \text{it is not that case that } FX \text{ falls under the concept } X, \text{ and} \\
(3) & \quad /\Pi X', \text{ where } X' \text{ is the concept instantiated just by } FX \text{ and every item whichinstantiates } X \text{ (i.e., } \forall x[X'x \equiv (Xx \lor x = FX)]; \text{ in set-theoretic terms, } X' \text{ is } (X \cup \{FX\}).
\end{align*}
\]

Intuitively, the idea is that the sub-concepts of \( P \) of which \( /\Pi \) holds have no maximal member.\(^8\) For any sub-concept \( X \) of \( P \) such that \( /\Pi X \), there is a proper extension \( X' \) of \( X \) such that \( /\Pi X' \).

This relativized notion of indefinite extensibility is quite robust, covering a lot of different situations. Below we give twelve examples. (The reader may choose to skip some at first reading.)

1. \( Px \Leftrightarrow x \text{ is a finite ordinal (or cardinal) number}; /\Pi X \Leftrightarrow \text{there are only finitely many } X\text{’s}; FX \text{ is the successor of the largest } X. \) So being a finite ordinal (or cardinal) is indefinitely extensible with respect to ‘finite’.

2. \( Px \Leftrightarrow x \text{ is a countable ordinal (i.e., countable well-ordering type)}; /\Pi X \Leftrightarrow \text{there are only countably many } X\text{’s}; FX \text{ is the successor of the union of the } X\text{’s}. \) So being a countable ordinal is indefinitely extensible with respect to ‘countable’.

3. In general, let \( \kappa \) be any regular cardinal number,\(^9\) and define \( Px \Leftrightarrow x \text{ is an ordinal smaller than } \kappa. \) \( /\Pi X \Leftrightarrow \text{there are fewer than } \kappa \text{-many } X\text{’s}; FX \text{ is the successor to the}

\(^8\) Say that \( P \) is weakly indefinitely extensible with respect to \( /\Pi \) if and only if for each sub-concept \( X \) of \( P \) such that \( /\Pi X \), there is an item \( t \) of type \( \tau \) such that

\[
\begin{align*}
(1) & \quad Pt, \\
(2) & \quad \text{it is not that case } Xt, \text{ and} \\
(3) & \quad /\Pi X', \text{ where } X' \text{ is the concept which applies to } t \text{ and to every item to which } X \text{ applies.}
\end{align*}
\]

The difference, of course, is that with the stronger notion characterized above, it is required that there be a function that gives the extra element \( t \). The strong notion is equivalent to the weak one if we assume a strong choice principle:

\[
\forall X\exists xR(X,x) \rightarrow \exists f\forall XR(X,fX),
\]

where \( x \) is a variable of type \( \tau \), \( X \) has the type of concepts of type \( \tau \) items, and \( f \) has the appropriate function-type. See note 2 above.

\(^9\) Recall that a cardinal \( \kappa \) is ‘regular’ if no set of size \( \kappa \) is the union of fewer than \( \kappa \)-many sets each of which is smaller than \( \kappa \). It follows from the axiom of choice that every infinite successor
union of the $X$’s. So, for each regular cardinal $\kappa$, the concept of ‘being an ordinal smaller than $\kappa$’ is indefinitely extensible with respect to ‘smaller than $\kappa$’.

A converse holds. A cardinal $\kappa$ is regular if and only if ‘being an ordinal smaller than $\kappa$’ is indefinitely extensible with respect to ‘smaller than $\kappa$’ using the indicated ‘successor of union’ function.

4. Let $\kappa$ be any infinite cardinal, and define $P_x$ iff $x$ is an ordinal smaller than $\kappa$. $\Pi X$ iff there are fewer than $\kappa$-many $X$’s; $FX$ is the smallest ordinal $\lambda$ such that $P_\lambda \& \neg X\lambda$. So, for each infinite cardinal $\kappa$, the concept of ‘being an ordinal smaller than $\kappa$’ is indefinitely extensible with respect to ‘smaller than $\kappa$’.

5. Let $\kappa$ be a strong inaccessible. $P_x$ iff $x$ is an ordinal smaller than $\kappa$; $\Pi X$ iff there are fewer than $\kappa$-many $X$’s. $FX$ is the powerset of the union of the $X$’s. So, for each strong inaccessible $\kappa$, the concept of ‘being an ordinal smaller than $\kappa$’ is indefinitely extensible with respect to ‘smaller than $\kappa$’ using the indicated ‘successor of union’ function. (Here, again, there is a converse.)

6. $P_x$ iff $x$ is a real number; $\Pi X$ iff there are only countably many $X$’s. Define $FX$ using a Cantorian diagonal construction. So being a real number is indefinitely extensible with respect to ‘countable’.

7. $P_x$ iff $x$ is (the Gödel number of ) a truth of arithmetic; $\Pi X$ iff the $X$’s are recursively enumerable. $FX$ is a Gödel sentence generated by the $X$’s, or the straightforward statement that the $X$’s are consistent. Then, if every member of $X$ is true of the natural numbers, then so is the sentence $FX$. And, of course, $FX$ is not one of the $X$’s. So being (the Gödel number of ) a truth of arithmetic is indefinitely extensible with respect to the property of being recursively enumerable.

As noted, Dummett first introduced the terminology of indefinite extensibility in his (1963). The only indefinitely extensible notion discussed there is the preceding, for which the case can be generalized as indicated in the previous section:

8. $P_x$ iff $x$ is the Gödel number of a truth of arithmetic; $\Pi X$ iff the $X$’s are arithmetic, i.e., iff there is a formula $\Phi(x)$ with only $x$ free such that $\Phi(x)$ iff $Xx$. $FX$ is a fixed point for $\neg \Phi(x)$: the Gödel number $n$ of a sentence $\Psi$ such that ($\Psi \equiv \neg \Phi(n)$) is provable in ordinary Peano arithmetic (and so true). So being (the Gödel number of ) a truth of arithmetic is indefinitely extensible with respect to the property of being arithmetic.

With even more generality, Tarski’s theorem is in effect that the notion of truth for any sufficiently rich language is indefinitely extensible with respect to concepts definable in that language. Another generalization is studied in recursive function theory:

9. Let $A$ be a productive set of natural numbers (see Rogers, 1967, 84). $P_x$ iff $x \in A$; $\Pi X$ iff $X$ is recursively enumerable. So, for each productive set $A$, the concept cardinal is regular. Given Choice, the smallest infinite non-regular cardinal is $\aleph_\omega$. It is common nowadays to identify ordinals with von Neumann ordinals and to identify cardinals with alephs, which, in this context, are von Neumann ordinals of minimal cardinality. That is, $\kappa$ is an aleph if $\kappa$ is a von Neumann ordinal and the cardinality of each $\lambda \in \kappa$ is less than that of $\kappa$. With these identifications, $P_x$ iff $x \in \kappa$. 

All Things Indefinitely Extensible 267
of being a member of \( A \) is indefinitely extensible with respect to the property of being recursively enumerable.

Notice that the set of Gödel numbers of arithmetic truths is productive.

Finally—and as they had better—the three notions invoked at the start to introduce the notion of indefinite extensibility also fit the present template:

10. \( P_x \iff x \) is an ordinal (or a von Neumann ordinal); \( \Pi X \iff \) each of the \( X \)'s is an ordinal and the \( X \)'s are themselves isomorphic to an ordinal (under the natural ordering). In other words, \( \Pi X \iff \) each \( X \) is an ordinal and the \( X \)'s have (or exemplify) a well-ordering type. (In still other words, \( \Pi X \iff \) the \( X \)'s are a set of ordinals.) \( FX \) is the successor of the union of the \( X \)'s. So being an ordinal is indefinitely extensible with respect to the property of being isomorphic to an ordinal (or exemplifying a well-ordering type).

This, of course, is just the Burali–Forti construction.

11. \( P_x \iff x \) is a set that does not contain itself; \( \Pi X \iff \) the \( X \)'s form a set, i.e., \( \exists y \forall x (x \in y \iff x \in X \}) \). \( FX \) is just the set of \( X \)'s: \( \{ x | X x \} \). So being a set that does not contain itself is indefinitely extensible with respect to the property of being, or constituting, a set.

12. \( P_x \iff x \) is a cardinal number; \( \Pi X \iff \) the \( X \)'s form a set (with a cardinal number). Given such an \( X \), take the union of a totality consisting of at least one set whose cardinality is in a set that has \( X \) (using Choice and Replacement); \( FX \) is the powerset of that. So being a cardinal number is indefinitely extensible with respect to the property of being, or constituting, a set.

Most of these instances of relativized indefinite extensibility are unremarkable. They do not, as far as they go, shed any philosophical light on the paradoxes. Our ultimate goal, of course, remains to define an unrelativized notion of indefinite extensibility, a notion that covers ordinal, cardinal, and set and at least purports to shed some light on the paradoxes, in the sense that the latter should emerge as somehow turning on the indefinite extensibility of the concepts concerned. So what next?

Three further steps are needed. Notice to begin with that the listed examples subdivide into two kinds. There are those where—helping ourselves to the classical ordinals—we can say that some ordinal \( \lambda \) places a lowest limit on the length of the series of \( \Pi \)-preserving applications of \( F \) to any \( X \) such that \( \Pi X \). Intuitively, while each series of extensions whose length is less than \( \lambda \) results in a collection of \( P \)'s which is still \( \Pi \), once the series of iterations extends as far as \( \lambda \) the resulting collection of \( P \)'s is no longer \( \Pi \), and so the ‘process’ stabilizes. This was the situation noted with numerically determinate expression of English and arithmetical truth, as discussed in preceding sections, and it is also the situation of all but examples 10, 11 and 12 listed above. In those three cases, by contrast, there is no ordinal limit to the \( \Pi \)-preserving iterations. With 10, this is obvious, since the higher-order property \( \Pi \) in that case just is the property of having a well-ordering type. Indeed, let \( \lambda \) be an ordinal. Then the first \( \lambda \) ordinals have the order type \( \lambda \) and so they have the property. The ‘process’ thus does not terminate or stabilize at \( \lambda \). With 11 and 12, we get the same result if we assume
that for each ordinal $\lambda$, a totality that has order type $\lambda$ is a set and has a cardinality. This is just the replacement principle invoked in the argument for the Russell conjecture in Section 10.2 above.

Let’s accordingly refine the relativized notion to mark this distinction. So first, for any ordinal $\lambda$ say that $P$ is up-to-$\lambda$-extensible with respect to $\Pi$ just in case $P$ and $\Pi$ meet the conditions for the relativized notion as originally defined but $\lambda$ places a limit on the length of the series of $\Pi$-preserving applications of $F$ to any sub-concept $\chi$ of $P$ such that $\Pi\chi$. Otherwise put, $\lambda$ iterations of the extension process on any $\Pi\chi$ ‘generates’ a collection of $P$’s which form the extension of a non-$\Pi$ sub-concept of $P$.

Next, say that $P$ is properly indefinitely extensible with respect to $\Pi$ just if $P$ meets the conditions for the relativized notion as originally defined and there is no $\lambda$ such that $P$ is up-to-$\lambda$-extensible with respect to $\Pi$. Finally, say that $P$ is indefinitely extensible (simpliciter) just in case there is a $\Pi$ such that $P$ is properly indefinitely extensible with respect to $\Pi$.

Our suggestion, then, is that the circularity involved in the apparent need to characterize indefinite extensibility by reference to Definite sub-concepts/collections of a target concept $P$ can be finessed by appealing instead at the same point to the existence of some species— $\Pi$— of sub-concepts of $P$/collections of $P$’s for which $\Pi$-hood is limitlessly preserved under iteration of the relevant operation. This notion is, naturally, relative to one’s conception of what constitutes a limitless series of iterations of a given operation. No doubt we start out innocent of any conception of serial limitlessness save the one implicit in one’s first idea of the infinite, whereby any countable potential infinity is limitless. Under the aegis of this conception, natural number is properly indefinitely extensible with respect to finite and so, just as Dummett suggests, indefinitely extensible simpliciter. The crucial conceptual innovation which transcends this initial conception of limitlessness and takes us to the ordinals as classically conceived is to add to the idea that every ordinal has a successor the principle that every infinite series of ordinals has a limit, a first ordinal lying beyond all its elements—the resource encapsulated in Cantor’s second number principle. If it is granted that this idea is at least partially—as it were, initial-segmentally—acceptable, the indefinite extensibility of natural number will be an immediate casualty of it. (Critics of Dummett who cannot see what he is driving at are presumably simply taking for granted the orthodoxy that the idea is at least partially acceptable.)

10.6 INDEFINITE EXTENSIBILITY AND THE PARADOXES

Roughly, then, $P$ is indefinitely extensible just in case, for some $\Pi$, any $\Pi$ sub-concept of $P$ allows of limitless $\Pi$-preserving enlargement. There seems to be nothing inherently paradoxical about this idea. So what is the connection with paradox—how is the indefinite extensibility of set, ordinal and cardinal linked with the classic paradoxes that beset those notions? The immediate answer is that in each of these cases there is powerful intuitive reason to regard $P$ itself as having the property $\Pi$. For example, in case $P$ is ordinal, and $\Pi X$ holds just if the $X$’s exemplify a well-order-type, it seems irresistible to say that ordinal itself falls under $\Pi$. After all, the ordinals
are well-ordered. But then the relevant principle of extension kicks in and dumps a
new object on us that both must and cannot be an ordinal—must because it cor-
responds, it seems, to a determinate order-type, cannot because the principle of exten-
sion always generates a non-instance of the concept to which it is applied. So we have the
Burali–Forti paradox.

The question, then, is what leads us to so fix our concepts of *set*, *ordinal*, and *car-
dinal* so that they seem to be indefinitely extensible with respect to Π’s which are,
seemingly, characteristic of those very concepts themselves? These remarks of Dum-
mett (1991, 315–16) suggest what we believe is the key insight:

to someone who has long been used to finite cardinals, and only to [finite cardinals], it seems
obvious that there can only be finite cardinals. A cardinal number, for him, is arrived at by
counting; and the very definition of an infinite totality is that it is impossible to count it . . . .
[But this] prejudice is one that can be overcome: the beginner can be persuaded that it makes
sense, after all, to speak of the number of natural numbers. Once his initial prejudice has been
overcome, the next stage is to convince the beginner that there are distinct transfinite cardinal
numbers: not all infinite totalities have as many members as each other. When he has become
accustomed to this idea, he is extremely likely to ask, ‘How many transfinite cardinals are
there?’ How should he be answered? He is very likely to be answered by being told, ‘You
must not ask that question’. But why should he not? If it was, after all, all right to ask, ‘How
many numbers are there?’, in the sense in which ‘number’ meant ‘finite cardinal’, how can it be
wrong to ask the same question when ‘number’ means ‘finite or transfinite cardinal’? A mere
prohibition leaves the matter a mystery. It gives no help to say that there are some totalities so
large that no number can be assigned to them. We can gain some grasp on the idea of a totality
too big to be counted . . . . but once we have accepted that totalities too big to be counted may
yet have numbers, the idea of one too big even to have a number conveys nothing at all. And
merely to say, ‘If you persist in talking about the number of all cardinal numbers, you will run
into contradiction’, is to wield the big stick, but not to offer an explanation.

What the paradoxes revealed was not the existence of concepts with inconsistent extensions,
but . . . indefinitely extensible concepts.

We have already noted that indefinite extensibility does not *per se* seem paradox-
ical, so the insight that Dummett is giving expression to is not well summarized by
the last two quoted lines. The insight is rather into the interconnection, in the case
in point, between the indefinite extensibility of *cardinal number* and the temptation
to say that the concept falls under—*ought* to fall under—the relevant Π. We get
the indefinitely extensible series of transfinite cardinals up and running in the first
place by insisting on one-one correspondence between concepts as necessary and suf-
cient for sameness, and hence existence, of cardinal number in general—not just in
the finite case—and then, under the aegis of that insistence, by bringing to bear the
Axioms of Union and Powerset, and then Cantor’s theorem. A conception of cardinal
embracing both the finite and the spectacular array of transfinite cases only arises in
the first place when it is taken without question that concepts in general—or at least
any that sustain determinate relations of one-one correspondence—have cardinal
numbers, identified and distinguished in the light of those relations. That is how the
intuitive barrier to the question, how many natural numbers are there, is overcome.
But then the lid is off Pandora’s box: for the intuitive barrier to the question, how
many cardinal numbers are there is overcome too. Cardinal, it seems, has to be both indefinitely extensible with respect to has a cardinal number and an instance of it.

It is straightforward to transpose this diagnosis to our paradigm, the ordinals, taken intuitively as the order-types of well-orderings. Consider an imaginary Hero (cf. Wright, 1998) being introduced to the ordinal numbers. Suppose that she has been introduced to the finite ordinals, but not the infinite ones. She wonders about the order-type of the finite ordinals, and realizes that she has no ordinal for this — yet. So she thinks that there is no ordinal of finite ordinals. But we tell her that the finite ordinals do indeed have an ordinal, just not one that she has encountered already. She thus meets $\omega$, and she formulates the notion of ‘countable ordinal’. Hero then learns about $\omega + 1$, $2\omega$, $\omega^\omega$, $\varepsilon_0$, etc. Perhaps she reads Cantor (1883), or a contemporary text in set theory. So now she naturally asks about the order-type of the countable ordinals, and she encounters the same problem. We tell her that the countable ordinals do have an ordinal — just not one that she has encountered already. So she learns about $\omega_1$. Hero is a quick study, and she recognizes the pattern: every initial segment of the ordinals has an order-type — just not one featuring in the segment itself, but rather the next one after all those. But now she notices that the ordinals themselves are well-ordered, and so she inquires after the order-type of all ordinals. This time it seems we have the option neither of telling her that the ordinals do indeed have an order-type — just not one among those she has encountered already — nor of denying that they have any order-type. She asked about the order-type of ALL ordinals, and since the ordinals are well-ordered, there ought to be one. But if this order-type exists, it too is an ordinal and must therefore occur among the ordinals whose collective order-type she asked about. Ordinal has been so explained to her as to be indefinitely extensible with respect to has instances which exemplify a well-order-type — and the nemesis is that there then seems no option but to allow that it is itself an instance of this $\Pi$.

We leave it to the reader to construct a narrative for Hero’s corresponding experience with the notion of set itself.

The three classic paradoxes of the transfinite, then, arise not with indefinite extensibility as such — at least, not if that is characterized as we have proposed — but with a particular twist taken by the examples concerned: cases where we unwittingly load a concept with a principle of indefinite extension whose trigger-concept — the relevant $\Pi$ — can be denied of the concept in question only by making an arbitrary exception to a connection — e.g. that well-ordered collections have order-types, that concepts which sustain relations of one-one correspondence have cardinals, that well-defined collections comprise sets — which is integral to the operation of the principle in question. Of course it may seem perverse to caption the making of an exception necessary to avoid contradiction as arbitrary. But, as Dummett said, intimidation is one thing, and explanation is another.

The situation we find ourselves in is one in which, in the view of Graham Priest (2002), we bump up against one of a number of ‘limits of thought’ — effectively, the limit involved in the attempt coherently to conceive of an absolutely limitless process of iteration. Abstractly put, the attempt involves successively thinking beyond anything that presents itself as a limit, always being ready to postulate a next element or
stage lying beyond what temporarily passes as a barrier—a new element of the same general category as everything that comes before but differing, of course, by virtue of being, in one or another respect, of a new species. (Cantor’s second number principle, see Section 10.11 above.) The ‘limit of thought’ is reached when we attempt to form a conception of the entirety—the whole array—of the elements or stages involved in the absolutely limitless iteration. If we accept that, this time, the array cannot be transcended, then it seems there is a limit after all and accordingly that we have not succeeded in conceiving of genuine iterative limitlessness. But this time we no longer have the option of postulating another instance of the generic category of object in question (ordinal, cardinal, set), because we are supposed to be dealing with ALL of them—or at least trying to.¹⁰

Priest himself suggests a dialetheic resolution: give up the law of non-contradiction, and allow, for instance, that ordinal both does and does not have an order-type. So the ordinals are indeed absolutely limitless and at the same time transcended in thought: there is an ordinal which succeeds all the ordinals (and is itself succeeded in turn by ordinals . . . ) For those of a nervous disposition who find this too strong to stomach, it may seem that the only option is to deny that there is an ordinal of all ordinals, a set of all sets, and a cardinal of all cardinalities. There are several options—see Section 10.11 below. That standard set-theory (ZFC) itself sanctions these denials does not of course make them principled.

10.7 A CONFESSION

Sua culpa. From the perspective just arrived at, the notion of ‘proper class’—made free use of in Shapiro (1991)—now looks quite illegitimate. Invoking proper classes is an attempt to do the very thing we are intuitively barred from doing—a fudge which attempts to allow both that set itself falls under various of the Π’s that trigger relevant principles of extension for sub-concepts/collections of it and that a new kind of ‘collection’ provides the corresponding values. The theorist who invokes proper classes thus confusedly thinks of himself as doing something analogous to Hero’s move from finite to countable ordinals. But he has forgotten that set is supposed to encompass the maximally general category of entities of the relevant kind. The point is trenchantly made by George Boolos, 1998a, 35–6:¹¹

Wait a minute! I thought that set theory was supposed to be a theory about all, ‘absolutely’ all, the collections that there were and that ‘set’ was synonymous with ‘collection’ . . . If one admits that there are proper classes at all, oughtn’t one to take seriously the possibility of an

¹⁰ Perhaps Cantor is giving expression to the strain here when he writes (1883, endnote 2): ‘we shall never reach a boundary that cannot be crossed; but we shall also never achieve even an approximate conception of the absolute. The absolute can only be acknowledged, but never known, not even approximately.’

¹¹ Boolos’s observation here is the main motivation behind his well-known pluralist reading of (monadic) second-order quantifiers—in effect, an attempt to make sense and secure some of the theoretical advantages of second-order set theory, without admitting special items for the higher-order quantifiers to range over.
iteratively generated hierarchy of collection-theoretic universes in which the sets which ZF recognizes [merely] play the role of ground-floor objects? I can’t believe that any such view of the nature of ‘∈’ can possibly be correct. Are the reasons for which one believes in [proper] classes really strong enough to make one believe in the possibility of such a hierarchy?

Similar strictures apply to what may be called ‘super-ordinals’—which would be well-orderings that are too big to be ordinals—and super-cardinals, which would be the sizes of proper classes. Cantor’s ‘inconsistent multiplicities’—considered as genuine objects—seem to be the same sort of thing as proper classes, and just as illegitimate. Zermelo was well advised to eschew all reliance on such ‘things’.¹²

10.8 NEO-LOGICISM, ANTI-ZERO AND FRIENDS

The first stage in the neo-logicist program is to develop arithmetic from Hume’s principle (HP):

$$\forall F \forall G [(\forall x : Fx = Nx : Gx) \equiv (F \approx G)],$$

where $F \approx G$ is an abbreviation of the second-order statement that there is a relation mapping the $F$’s one-to-one onto the $G$’s. In both Frege (1884) and Wright (1983), the opening quantifiers are unrestricted. This seems to be well motivated. One insight that underlies both Frege’s own treatment and neo-logicism is the universal applicability of arithmetic (Frege, 1884, §14). So long as one has objects, one can count them: arithmetic is applicable to absolutely any objects. So a principle governing identity of cardinal number in general should be formulated in such a way as to embrace (concepts of) objects of absolutely any sort (see Wright, 1998, 356–7).

Well motivated or not, Boolos (1997) launched an intriguing criticism of neo-logicism on the basis of this point. It follows from HP that self-identical has a cardinal number. This would be the number of all objects whatsoever—dubbed ‘anti-zero’ by Wright. Similarly, HP entails that there is a number of all cardinal numbers, a number of all ordinal numbers, and a number of all sets. Boolos observes that prima facie, this presents a conflict with ordinary Zermelo–Fraenkel set theory:

[Is there such a number as [the number of all objects whatsoever?] According to [ZF] there is no cardinal number that is the number of all the sets there are. The worry is that the theory of number [based on HP] is incompatible with Zermelo–Fraenkel set theory plus standard definitions.

(Boolos, 1997, 260)

A first reaction to the objection is that it involves an illicit commutation. Boolos writes ‘According to [ZF] there is no cardinal number that is the number of all the sets there are’, but all that is justified is something like the weaker ‘It is not the case that according to [ZF] there is a cardinal number that is the number of all the sets there are.’ ZF does not countenance proper classes, to be sure, and thus it does not assign

¹² See the passage quoted at the end of Section 10.2 above.
numbers to such items. But ZF does not deny the existence of such large ‘numbers’. As a first-order theory, it cannot even formulate the question. One would not criticize standard Peano arithmetic for failing to recognize real numbers, so why fault ZF for failing to recognize numbers of some (proper class sized) concepts?

However, as we interpret it, Boolos’s criticism of HP runs deeper than this, and is of a piece with his rejection of proper classes. The first-order variables of set-theory are supposed to range over every set-like object there is. If proper classes are set-like they should have been included in the range of the variables of set theory. With the axiom of choice, it is plausible to hold that a concept has a size only if it is equinumerous with a cardinal. If we reject proper classes, then cardinals are sets, and only sets have cardinals.

Wright (1999, 12–13) himself gives an independent argument against anti-zero, arguing that the cardinality operator is properly restricted to concepts that are sortal and that self-identical is not one of them. That point, however, offers no treatment of Boolos’s objection in relation to ordinal, cardinal, and set. For these cases Wright granted the force of Boolos’s objection, allowing ‘the plausible principle . . . that there is a determinate number of F’s just provided that the F’s compose a set’, and observing that ‘Zermelo–Fraenkel set theory implies that there is no set of all sets. So it would follow that there is no number of all sets’. On the same count, there is no number of all ordinals and no number of all cardinals—contra the straightforward reading of HP. Wright’s response is further to restrict the second-order variables in HP, so that some sortal concepts do not have numbers, and to invoke the notion of indefinite extensibility for this purpose. He writes:

I do not know how best to sharpen [the notion of indefinite extensibility] . . . Dummett could be [wrong about some of his claims concerning the notion but still] . . . emphasizing an important insight concerning certain very large totalities—ordinal number, cardinal number, set, and indeed ‘absolutely everything’. If there is anything at all in the notion of an indefinitely extensible totality . . . one principled restriction on Hume’s Principle will surely be that [cardinal numbers] not be associated with such totalities.

(Wright, 1999, 13–14)

Thus, Wright suggests that the second-order variables in HP be restricted to sortal concepts that are Definite. Wright’s programmatic suggestion provoked Shapiro (2003a). If we can restrict HP in the way Wright suggests—to avoid saying that there is a number of all ordinals, a number of all sets, etc.—then why not restrict Basic Law V similarly, and perhaps resurrect set theory along neo-logicist lines? Indeed, if P is a Definite concept, then the extension of P is just the set of P’s. No danger of contradiction there. Of course, to develop this project, we need a robust articulation of indefinite extensibility. We consider this direction further in Section 10.10 below.

In response to Wright’s proposal, Peter Clark (2000) argued that the best candidate for ‘not indefinitely extensible’ is ‘set sized’, where ‘set’ is the notion given by Zermelo–Fraenkel set theory. That is, Clark argues that ‘Definite’ just means something like ‘equinumerous with a member of the iterative hierarchy’. So if the notion of indefinite extensibility is indeed needed for the neo-logicist program, then that program is hopeless. It requires that we articulate the iterative hierarchy before we can
give the proper foundation even for arithmetic—before we can even so much as state Hume’s Principle in full generality.

If our proposals above about the proper characterization of indefinite extensibility are accepted, Clark’s objection has been answered. It may prove to be, as he suggests, that the extension of Definite coincides with that of set-sized. But that is not the direction in which to seek a proper characterization of Definite or its contrary. Rather, to repeat, an indefinitely extensible concept $P$ is one such that, for some $\Pi$, any $\Pi$ subconcept of the original allows of limitless $\Pi$-preserving intra-$P$ enlargement. There is no implicit appeal to the notion of set. It is true that, as we stressed above, the conception of limitlessness is parametric in the characterization, and one construction of it will run in harness with the ZF treatment of the ordinals. But that is (one) working out of the notion of indefinite extensibility, rather than something which belongs with the kind of explanation that would have to be given before a restriction of HP—or Basic Law V, for that matter—to Definite concepts could be presumed intelligible. And of course however it works out, finite concepts will pass the test. So at least the neo-logicist treatment of arithmetic is safe.

10.9 INDEFINITE EXTENSIBILITY, LIMITATION OF SIZE AND THE TRUE INFINITE—HOW MANY IS ‘TOO MANY’?

The principal thesis of Russell’s (1906) limitation of size conception of sets is that ‘there [is] (so to speak) a certain limit of size which no [set] can reach; and any supposed [set] which reaches or surpasses this limit is . . . improper . . . , i.e., is a non-entity’. Someone in sympathy with the tendency of our discussion so far will find it hard to take Russell’s expression of his point literally. Those of us trained in set theory will in any case find it natural to think of ‘size’ as something only sets have, following Cantor, Zermelo, et al. If a ‘collection’ is not a set, then it is nothing, has no size at all, and so can’t be ‘too big’. Indeed Russell himself speaks of ‘non-entity’. Moreover if Wright’s (1999) suggestion just rehearsed is accepted, indefinitely extensible concepts will determine neither sets nor cardinals; so there will be no size(s) for the ‘non-entities’ to have.

From the present point of view, the solid core to the suggestion that sets are subject to limitation of size is nothing but the thesis that indefinitely extensible concepts do not determine sets. What is striking is that Russell (1906, 153–154), for his part, dismisses the limitation-of-size conception (in favor of the no-class theory) almost as soon as he raises it, and that the reason he gives for doing so is absolutely consistent with and germane to this proposed interpretation of it. He writes:

A great difficulty of this theory is that it does not tell us how far up the series of ordinals it is legitimate to go. It might happen that $\omega$ was already illegitimate: in that case all proper [sets] would be finite . . . . Or it might happen that $\omega^2$ was illegitimate, or $\omega^\omega$, or $\omega_1$ or any other [limit] ordinal . . . [O]ur general principle does not tell us under what circumstances [a concept is Definite].

It is no doubt intended by those who advocate this theory that all ordinals should be admitted which can be defined, so to speak, from below, i.e., without introducing the notion of the
whole series of ordinals. Thus, they would admit all of Cantor’s ordinals, and they would only avoid admitting the maximum ordinal. But it is not easy to state such a limitation precisely: at least I have not succeeded in doing so.

It is obvious, of course, that anyone who wishes to propose a set theory developed in terms of ‘limitation of size’ must face the conceptual problem of delimiting just how ‘many’ objects a concept must apply to, in order for it to rank as inadmissibly big. But Russell’s objection, re-expressed in the light of the interpretation of size-limitation latterly proposed, comes to the concern that, without some independent and well-motivated grip on the idea of limitless iteration, we have no principled characterization of which concepts determine sets. Since limitless iteration is iteration without ordinal limit, it is not too far off the mark to express the point by wondering ‘how far up the series of ordinals it is legitimate to go’. But a better expression of the question would be: what structure should we attribute to the series of ordinals?

Dummett (1991, 317) himself writes that the ‘principle of extendibility constitutive of an indefinitely extensible concept is independent of how lax or rigorous the requirement for having a [D]efinite conception of a totality is taken to be, although that will of course affect which concepts are acknowledged to be indefinitely extensible’. That is exactly the same concern if we take it that one has such a Definite conception only if the ‘construction’ of the totality in question has an ordinal limit.

The issue is critical. We can sharpen our feel for it if we consider philosophers and mathematicians at two polar extremes. At the conservative end, Dummett evidently sympathizes with—and in some passages seemingly adopts—Russell’s intendedly flippant suggestion that even $\omega$ is too big to be Definite. In other words, Dummett claims that the concept of being a finite ordinal is already indefinitely extensible—that the finite ordinals have no ordinal limit. The structure of the ordinals—all the Definite ordinals—is that of a (misleadingly termed!) $\omega$-sequence. He notes that it is common for mathematicians to concede that concepts like set and ordinal (as normally liberally understood) are indefinitely extensible, but most hold that domains like the natural numbers and the real numbers are perfectly Definite. Dummett argues that this last belief is ungrounded:

We have a strong conviction that we do have a clear grasp of the totality of natural numbers; but what we actually grasp with such clarity is the principle of extension by which, given any natural number, we can immediately cite one greater than it by 1. A concept whose extension is intrinsically infinite is thus a particular case of an indefinitely extensible one. Assuming its extension to constitute a [D]efinite totality . . . may not lead to inconsistency; but it necessarily leads to our supposing that we have provided definite truth-conditions . . . for statements that cannot legitimately be so interpreted.

(Dummett, 1991, 318, see also 1993, 442–3)

The last remark asserts a connection between the indefinite extensibility of a concept and issues concerning determinacy of sense for certain kinds of statements concerning its instances—par excellence, quantified statements; an issue which takes us to the heart of the agenda for the present volume and to which we shall shortly turn. But in its general outline, though not of course in respect of that claimed connection,
Dummett’s stance belongs to the tradition initiated by Aristotle and elegantly represented by Leibniz.¹³

It could . . . well be argued that, since among any ten terms there is a last number, which is also the greatest of those numbers, it follows that among all numbers there is a last number, which is also the greatest of all numbers. But I think that such a number implies a contradiction . . . When it is said that there are infinitely many terms, it is not being said that there is some specific number of them, but that there are more than any specific number.

(Letter to Bernoulli, Leibniz 1863, III 566, translated in Levey 1998, 76–7, 87)

. . . we conclude . . . that there is no infinite multitude, from which it will follow that there is not an infinity of things, either. Or [rather] it must be said that an infinity of things is not one whole, or that there is no aggregate of them.


Yet M. Descartes and his followers, in making the world out to be indefinite so that we cannot conceive of any end to it, have said that matter has no limits. They have some reason for replacing the term ‘infinite’ by ‘indefinite’, for there is never an infinite whole in the world, though there are always wholes greater than others ad infinitum. As I have shown elsewhere, the universe cannot be considered to be a whole.

(Leibniz, 1996, 151)

For Leibniz, the infinite just is limitlessness; no actual infinities exist. The only intelligible notion of infinity is that of potential infinity—the transcendence of any limit.

For witnesses of the other polar extreme, we turn to the ultra-liberals, Cantor and Zermelo. In a sense, Cantor accepted the conception of the true infinite as pure limitlessness. His achievement, for those who believe in it, was to have discovered the rich ‘paradise’ of limits beyond the finite: the realm of transfinite ordinals and cardinals. Cantor’s belief in the actuality of the transfinite is supported by appeal to the alleged instability of the potentialist conception, sometimes in theological terms:

the potential infinite is only an auxiliary or relative (or relational) concept, and always indicates an underlying transfinite without which it can neither be nor be thought.

That an ‘infinite creation’ must be assumed to exist can be proved in many ways . . . One proof stems from the concept of God. Since God is of the highest perfection one can conclude that it is possible for Him to create a transfinitum ordinatum. Therefore, . . . we can conclude that there actually is a created transfinitum.

(Cantor, 1887, 391, 400)

Responding to the suggestion that the actual infinite is unintelligible, ‘that we with our restricted being are not in a position to actually conceive the infinitely many individuals . . . belonging to the set . . . in one intuition’, Cantor replies:

But I would like to see that man who, for instance, can form the idea distinctly and precisely in one intuition of all the unities in the finite number ‘thousand million’, or some even smaller numbers. No one alive today has this ability. And yet we have the right to acknowledge the

¹³ Thanks to Roy Cook for suggesting these references.
finite numbers, however great, as objects of discursive human knowledge, and to investigate their concepts scientifically. We have the same right also with respect to the transfinite numbers.

(Cantor, 1887, 402)

With these remarks, Cantor challenges his opponents to delimit a principled alternative to strict finitism—a finitism going no further than our actual practical limitations of intuition and understanding—that does not also sanction the transfinite.¹⁴ Grasping the notion of a large finite set already requires idealization. What reason is there to limit the idealization to the arbitrarily large but still finite?

We are invited to answer ‘none’. But if we do, where should we locate the true infinite—as opposed to the merely transfinite? Recall Cantor’s definition (from (1883, note 1)): ‘By a ‘manifold’ or “set” I understand any multiplicity which can be thought of as one, i.e., any aggregate of determinate elements which can be unified into a whole by some law’. The idea here is that if it is (merely) consistent for some objects to be a ‘unity’, then it is a unity—there is an actual set that contains just those objects. This principle underlies Cantor’s entire project. Once we cast off the shackles of potentialism, only consistency is allowed to put a brake on the exhilarating rush beyond.

Almost half a century later, Zermelo (1930) articulates a version of second-order ZFC with urelements, in pretty much its contemporary form, and he freely discusses models of the axiomatization. If a model of the theory lacks urelements, then it is isomorphic to a rank \( V_\kappa \) in which \( \kappa \) is a strong inaccessible. He proposes (1930, 1233) an axiom stating the existence of ‘an unbounded sequence’ of such models. Each such model \( V_\kappa \) has subsets (like \( \kappa \), the collection of ordinals in the model) which are not members of the model.¹⁵ However,

[w]hat appears as an ‘ultra-finite non- or super-set’ in one model is, in the succeeding model, a perfectly good, valid set with both a cardinal number and an ordinal type . . . To the unbounded series of Cantor ordinals there corresponds a similarly unbounded . . . series of essentially different set-theoretic models. Scientific reactionaries and anti-mathematicians have so eagerly and lovingly appealed to the ‘ultra-finite antinomies’ in their struggle against set theory. But these are only apparent ‘contradictions’, and depend solely on confusing set theory itself . . . with individual models representing it . . . The two polar opposite tendencies of the thinking spirit, the idea of creative advance and that of collection and completion, ideas which also lie behind the Kantian ‘antinomies’, find their symbolic reconciliation in the transfinite number series based on the concept of well-ordering. This series reaches no true completion in its unrestricted advance, but possesses only relative stopping-points, just those [strong inaccessibles] which separate the higher model types from the lower. Thus the set-theoretic ‘antinomies’, when correctly understood, do not lead to a cramping and mutilation of mathematical science, but rather to an, as yet, unsurveyable unfolding and enriching of that science.

In present terms, then, Zermelo’s proposal is that the series of models of second-order ZFC—and so the series of strongly inaccessible cardinals—is itself indefinitely

¹⁴ For an extensive, insightful, and compelling account of Cantor’s views here, see Hallett (1984), especially §§1.2 – 1.3.

¹⁵ To get the idea, consider someone, say our friend Hero, whose entire universe is one of these models (a set that we, from the outside, see as a \( V_\kappa \)). For Hero, some of (what we see as) subsets of \( V_\kappa \), such as \( \kappa \) itself, are indefinitely extensible, non-entities, whatever. But once Hero recognizes the next model after \( V_\kappa \), she sees see that those totalities are perfectly good sets.
extensible. Each strong inaccessible is a Definite collection, but any set of inaccessibles gives rise to further, larger, strongly inaccessible sets, cardinals, and ordinals. So there is no set of all such models or all such cardinals.

Zermelo proposed ‘the existence of an unbounded sequence of’ [inaccessible ranks] as a new axiom of “meta-set theory”. In effect, the new principle states that for each ordinal \( \alpha \), there is a unique inaccessible cardinal \( \kappa_\alpha \). This, of course, is what the Russell conjecture (from Section 10.2 above) would predict, concerning the statement that the strong inaccessible cardinals are themselves indefinitely extensible.

It is common for set-theorists to use language like Zermelo’s. We are not so bold as to suggest, for even one minute, that we—or they—do not know what they mean. Nevertheless, this is an almost literally breathtaking marginalization of the scope of the Aristotelian infinite. Each inaccessible is an actual infinity; the only potential infinity is the ‘collection’ of all strong inaccessibles. The ‘process’ of generating more strong inaccessibles has absolutely no limit, not even an inaccessibly large one. (Of course, this stretches the notion of ‘process’ even further, well beyond recognition; it also constitutes an exception to Cantor’s second number principle.)

As we saw, Russell (1906, 154) suggested that the underlying idea is that the actual infinite extends as far up the iterative hierarchy as it consistently can. Everything infinite in the hierarchy is actually infinite. It is hardly a substantial observation that the perspective of Cantor and Zermelo is ontologically extravagant! (Boolos, 1998b). But the real concern is that it is, au fond, unprincipled. ‘Keep going until you run into contradiction’ is not a principle but a refusal to be disturbed by the lack of one. The statement is intuitively insufficient for just the same reason that it is an intuitively insufficient response to the semantic paradoxes to claim simply that it is analytic of the concept of truth that the Equivalence Scheme (\( P \iff \text{true}(P) \)) holds in all cases save where it leads to contradiction. The fact is that one does run into contradiction unless a brake is put on proceedings somewhere, and one wants a principled account of where—something which marks off the exceptions as an independent species and provides an explanation of why a treatment which overlooks their distinctive character could be expected to lead to trouble. Yet the polar opposite, thoroughgoing conservative, Aristotelian view that all infinities are indefinitely extensible seems far too restrictive—it denies us what most will find no cause to doubt to be perfectly clear conceptions of the structures distinguished in at least the ‘early’ stages of the Cantorian transfinite. But then where, and how, is the line to be drawn? Dummett’s (1963) original claim that arithmetical truth is indefinitely extensible is perhaps consistent with setting the ‘limit’ a bit higher than \( \omega \). We might admit computable (or recursive) countable ordinals, for example. But Dummett’s ‘limit’ must be well short of \( \omega_1 \), and thus of the real numbers. We presume that most mathematicians have come to accept at least a few uncountable, Definite infinities—say the collection of all real-valued functions. But how is this issue to be adjudicated? On what grounds?

To summarize: we are proposing that the Aristotelian infinite is to be understood as the indefinitely extensible. The two polar views are then respectively that there is no Definite infinite—the infinite is the indefinitely extensible—and that the Definite infinite extends as far as it may consistently be taken to do. The first is ‘principled’ enough—it amounts to an across-the-board repudiation of Cantor’s Second
Principle: no collection of numbers, sets or ordinals which has no largest member has a limit: a smallest number, set or ordinal greater than any in the original. Unbridled liberalism, by contrast, which holds that the only exceptions to the principle are those dictated by mere consistency, is open to the charge of casuistry. A natural standpoint—‘natural’ at least in the anthropological sense that it seems to come naturally to many—is to want something in between: to want to be in position to make principled use of Cantor’s second principle, rather than simply reject it, while able to corral and account for the exceptions.

Hitherto we have understood the Definite simply as that which is not indefinitely extensible, that which is not limitless. But the parametric role of the concept of limitlesslessness in the characterization of indefinite extensibility means that there is no hope of progress past the impasse latterly outlined unless we can infuse Definiteness with some additional independent operational content, sufficient to provide a criterion for which are the Definite infinite totalities. But how is this to be achieved? As noted, Dummett concedes for his part that we can consistently think of \(\omega, \omega_1\), etc., as Definite, but argues that this simply begs the question. In (1991, 317–8), he seemed resigned to settling for a stand-off:

One reason why the philosophy of mathematics appears at present to be becalmed is that we do not know how to accomplish the task at which Frege so lamentably failed, namely to characterise the domains of the fundamental mathematical theories so as to convey what everyone, without preconceptions, will acknowledge as a definite conception of the totality in question: those who believe themselves already to have a firm grasp of such a totality are satisfied with the available characterizations, while those who are sceptical of claims to have such a grasp reject them as question-begging or unacceptably vague. An impasse is thus reached, and the choice degenerates into one between an act of faith and an avowal of disbelief, or even between expressions of divergent tastes. Moreover, the impasse seems intrinsically impossible of resolution; for fundamental mathematical theories, such as the theory of the natural numbers or the theory of real numbers, are precisely those from which we initially derive our conceptions of different infinite cardinalities, and hence no characterisation of their domains could in principle escape the accusation of circularity

But just a couple of years later, Dummett can be found arguing that the burden of proof lies with his opponent, one who claims that there are Definite infinite totalities:

It might be objected that no contradiction results from taking the real numbers to form a definite totality. There is, however, no ground to suppose that treating an indefinitely extensible concept as a definite one will always lead to inconsistency; it may merely lead to our supposing ourselves to have a definite idea when we do not

The totality of natural numbers contains what, from our perspective, are enormous numbers, and yet others [relative] to which those are minute, and so on indefinitely; do we really have a grasp of such a totality?

A natural response is to claim that the question has been begged. In classing real number as an indefinitely extensible concept, we have assumed that any totality of which we can have a definite conception is at most denumerable; in classing natural number as one, we have assumed that such a totality will be finite. Burden-of-proof controversies are always difficult to resolve; but, in this instance, it is surely clear that it is the other side that has begged the question.
It is claiming to be able to convey a conception of the totality of real numbers, without circularity, to one who does not yet have it... [The beginner] does not assume as a principle that any totality of which it is possible to form a definite conception is at most denumerable: he merely has as yet no conception of any totality of higher cardinality... The fact is that a concept determining an intrinsically infinite totality—one whose infinity follows from the concept itself—simply is an indefinitely extensible one.

(Dummett, 1993, 442–3)

Dummett’s stance in this passage is apparently that one should refuse to accept that the natural numbers or the real numbers—let alone the power set of the continuum, or totalities associated with inaccessible, or supercompact cardinals—are Definite until it has been shown, without circularity and without begging any questions, that Definite conceptions are possible of such totalities.

But Dummett here sets his opponents a task of which he has provided no adequate characterization. What should count as accomplishing it? We have some idea, perhaps, what is it to possess a Definite conception of a totality modulo a given assumption about the extent of the ordinals: it is to possess an (adequately clear?) conception of a totality which is either finite or only up-to-λ extensible for some ordinal λ recognized by that assumption. But how does one show that one has a Definite conception in the more robust sense now demanded—a conception, in particular, that would let one justify assumptions about the extent of the ordinals themselves? In general, argument that a burden of proof lies with an opponent is more persuasive when accompanied by the courtesy of an explanation of what exactly is the content of that which has to be proved, and what will be accepted as amounting to proof of it!¹⁶

Fortunately, there is an alternative to a resolute adherence to either of the polar views, even without philosophical progress on the main issue: it is to think of modern arithmetic, analysis, and set theory as exploring the consequences of a working hypothesis that the natural numbers, the real numbers, and other very large, infinite totalities allow of coherent conception as Definite. We cannot—yet, and maybe never will be able to—justify these hypotheses from first principles in the Philosophical Theory of Understanding, but we do not have to. Since Gödel, we have become used to flying without a safety net. In this way, the mathematician can in good conscience rest content with the theories in question, even without possessing the justification whose want the philosopher laments. Pro tem, he may let them stand or fall on the basis of the fruits they bear, wherever these fruits may lie.

In an insightful article on Cantor, Tait (2000, §4) writes in a similar spirit:

It was Cantor’s construction of the system of transfinite numbers... that opened a Pandora's box of foundational problems in mathematics, namely, the question of what cardinal numbers there are. One can, in a way, understand the resistance to Cantor’s ideas on the part of the mathematical law-and-order-types—in the same way that one can understand the Church terrorizing the elderly Galileo: in defense of a closed, tidy universe. In that respect, Hilbert’s

¹⁶ Dummett’s claims concerning the burden-of-proof are no doubt consonant with the sometime verificationist and foundationalist elements in his metaphysics; but one might well indeed despair of progress, we submit, if the issue cannot be resolved without entry into the chamber of such vexed debates.
reference to Cantor’s ‘Paradise’ is ironic: it was the Kroneckers who wanted to stay in Paradise and it was Cantor who lost it for us—bless him.

there are many mathematicians who will accept the Garden of Eden, that is, the theory of functions as developed in the nineteenth century, but will, if not reject, at least put aside the theory of transfinite numbers, on the grounds that it is not needed for analysis. Of course, on such grounds, one might also ask what analysis is needed for, and if the answer is basic physics, one might then ask what that is needed for. When it comes to putting food in one’s mouth, the ‘need’ for any real mathematics becomes somewhat tenuous. Cantor started us on an intellectual journey. One can peel off at any point, but no one should make a virtue of doing so.

Once again, it seems overly restrictive to follow Dummett and reject any Definite infinite totalities, or even any uncountable infinite totalities. Cantor’s paradise is too enticing, and fruitful. It is ad hoc to look for a ‘boundary’ or ‘limit’ intermediate between the staunch conservative, ‘law-and-order’ extreme and Cantor’s and Zermelo’s admission of everything. Any proposed place to stop the expansion of the actual infinite, say at $2\omega$, $\omega_1$, or the ninth strongly inaccessible, is artificial. Why stop there? As Tait (ibid., 284) puts it:

Pandora’s box is indeed open: Under what conditions should we admit the extension of a property of transfinite numbers to be a set—or equivalently, what transfinite numbers are there? No answer is final, in the sense that, given any criterion for what counts as a set of numbers, we can relativize the definition of $\Omega$ to sets satisfying that criterion and obtain a class $\Omega'$ of numbers. But there would be no grounds for denying that $\Omega'$ is a set: the preceding argument that $\Omega$ is not a set merely transforms in the case of $\Omega'$ into a proof that $\Omega'$ does not satisfy the criterion in question. So . . . we can go on. In the foundations of set theory, Plato’s dialectician, searching for the first principles, will never go out of business.

Suppose, for example, that someone—a less moderate minimalist—tries to define an ordinal to be a well-ordering-type that is ‘accessible’ (i.e., not strongly inaccessible). Then she can conclude that all ordinals are accessible, and be done with it. But, in the spirit of Cantor and Zermelo, it seems better to claim that we now have a reason to believe in an inaccessible ordinal: the ‘totality’ of our friend’s ordinals is itself an inaccessible well-ordering type, and thus an inaccessible ordinal.

Nevertheless, we must add the annoying reminder that one must ‘peel off’ Cantor’s journey at some point, to use Tait’s metaphor. The laws that drive the journey—Cantor’s two principles of generation (Section 10.1 above)—cannot be exceptionless, on pain of paradox. But where do we stop?

10.10 NEO-LOGICIST SET THEORY

We now return briefly to a question deferred from Section 10.8—the question whether the diagnosis of the paradoxes as, in effect, due to a failure to exclude indefinitely extensible concepts from within the range of legitimate set abstraction is in any way helpful when it comes to the attempt to develop a reasonably powerful set-theory along neo-logicist lines. ‘Reasonably powerful’, we shall take it, should
involve provision of resources sufficient not just for the reconstruction of classical arithmetic and analysis but for recovery of at least all the less than inaccessibly infinite sets in the standard iterative hierarchy and their standard mathematical treatment. It would be a significant coup for neo-logicism if such a theory could be based just on second-order logic and otherwise acceptable abstraction principles—in effect, if a neo-logicist reconstruction of ZFC could be accomplished whose epistemological basis was in no interesting way different from that of Frege Arithmetic—arithmetic based on second-order logic and Hume’s Principle. The purpose of this section is to indicate why, as we believe, this prospect is no closer.

Specialized to the case where courses-of-values are extensions of concepts—or we may as well say, sets—Frege’s inconsistent Basic Law V may be represented thus:

\[(\forall F)(\forall G)(\{x : Fx\} = \{x : Gx\} \equiv (\forall x)(Fx \equiv Gx))\],

a principle that both encapsulates the extensionality of sets and associates every concept with its own set. Frege’s own reaction to the paradox was to qualify extensionality: we can still count it, he suggested, as sufficient but not necessary in order for the set of \(F\)’s to be identical with the set of \(G\)’s that \(F\) and \(G\) be coextensive—rather what is necessary and sufficient is that \(F\) and \(G\) hold of exactly the same items save possibly the respective sets themselves. Frege’s proposal is obviously philosophically hopeless from the neo-logicist point of view: its formulation involves not just impredicative quantification on the right hand side of the abstraction over the newly introduced abstracts—something which is epistemologically controversial, of course, but which neo-logicism is anyway committed to arguing can be acceptable—but explicit mention of them using the canonical notation which the abstraction is supposed to explain. Worse, it lets us identify a pair of objects in circumstances where one has a property which the other lacks! Worst of all, it is also, as Frege must have rapidly realized, technically hopeless since inconsistent in any domain of more than one object.¹⁷

If there is to be a progressive neo-logicist modification of Law V, it must come, it would appear, from tweaking not its extensionality component, but the comprehension it affords: not all concepts can be permitted to determine (extensionally individuated) sets. In his (1986), George Boolos put forward one naturally and ingeniously conceived proposal: restrict Law V to concepts which, in keeping with the tradition of limitation of size, are ‘small’, with the complement of smallness glossed as ‘equinumerous with the universe’—a restriction which may be defined using only second-order logical restrictions.

There are two ways one can write in the restriction. One is to have it as antecedent in a universally quantified conditional whose consequent is Law V less its initial higher-order quantifiers:

\[(\forall F)(\forall G)[(smallF & smallG) \rightarrow (\{x : Fx\} = \{x : Gx\} \equiv (\forall x)(Fx \equiv Gx))],\]

but this has the drawback that one cannot decide whether a set is well-behaved—is individuated extensionally—before one knows how big the universe is. For example,

¹⁷ See Quine (1955) and Geach (1956). The first to remark on this seems to have been Lesniewski (see Sobociński, 1949). A useful diagnostic discussion is Resnik (1980, 211–20).
even if \(x \neq x\) is treated as a small concept as a matter of courtesy, as it were, one will struggle to distinguish \(\{x : x \neq x\}\) from \(\{x : x = x \neq x\}\) by the proposed principle unless one has it independently that the universe consists of more than one thing.

Hence the direction which Boolos actually took:

New \(V\)

\[(\forall F)(\forall G)(\{x : Fx\} = \{x : Gx\} \equiv ((\text{non-small} F \& \text{non-small} G) \lor (\forall x)(Fx \equiv Gx)))\]

The right hand side is still an equivalence relation, since co-extensiveness is a congruence for \textit{non-small}. Every concept gets an extension, but those bijectable with the universal concept all get the same extension, whether or not co-extensive. According to New \(V\), \(x \neq x\) and \(x = \{x : x \neq x\}\) are assigned distinct extensions, just as they should be, since they are not co-extensive and cannot both be non-small.

As Boolos shows, New \(V\) encompasses a surprising range of standard set-theoretic axioms. A ‘set’ is the extension of a small concept. Extensionality, Null Set, Pairs, Choice, Separation (\textit{Aussonderung}), and Replacement hold on all sets; and Union and Foundation hold on so-called ‘pure sets’, those built up, hereditarily, from the empty set. New \(V\) provides for a satisfactory theory of the hereditarily finite pure sets and a foundation for arithmetic. But as Boolos shows, it fails to provide Infinity—to ensure that there is any well-behaved (i.e. small) infinite set—and, even if we collateral assume that there is such a set, New \(V\) fails to ensure that it has a well-behaved power set. So Paradise is postponed.

Shapiro (2003a) launches an investigation of what, in general terms, may be expected foundationally from abstractions of the general pattern that New \(V\) exemplifies:

\[(\forall F)(\forall G)(\{x : Fx\} = \{x : Gx\} \equiv ((\text{bad} F \& \text{bad} G) \lor (\forall x)(Fx \equiv Gx)))\]

Obviously the paradoxes entailed by original Law \(V\) will now discharge themselves into proofs that the relevant concepts—and in particular, \textit{not-a-member-of-itself}, and (subject to appropriate definitions) \textit{ordinal} and \textit{cardinal}—are all bad, thus harmlessly marginalizing the concepts concerned.

Very well. So here is the salient question: how would matters turn out if—as would have been entirely consonant with our reflections in the early part of the preceding section—Boolos had proposed not \textit{non-smallness} but \textit{indefinite extensibility} as the appropriate reading of \textit{bad}?—if he had proposed what we may dub \textit{Indefinite \(V\)}:

\[(\forall F)(\forall G)(\{x : Fx\} = \{x : Gx\} \equiv ((\text{indef. extensible} F \& \text{indef. extensible} G) \lor (\forall x)(Fx \equiv Gx)))\]

There is one immediate potential point of improvement over New \(V\). On pain of the Burali–Forti paradox, \textit{ordinal} has to be bad under whatever reading. So if \textit{bad} is \textit{equinumerous with the universe}, as with New \(V\), then there is a bijection between the ordinals and everything that there is, and the universe is consequently well-ordered (see Shapiro and Weir, 1999). So the existence of a global well-ordering is a consequence of New \(V\). It follows that the non-abstracts are well-ordered, and that contravenes Wright’s (1997, pp. 230–3) conservativeness requirement for abstraction principles, that—roughly—(a priori) acceptable abstractions should not entail new
results about the old ontology. This problem will not affect Indefinite V, however, unless it can be shown that all indefinitely extensible concepts are bijectable with each other. The strongest result we have in the vicinity is Russell’s Conjecture, that every indefinitely extensible concept sustains an injection of the ordinals, not a bijection.

If indefinite extensibility is a matter of ‘size’—that is, if any collection equinumerous to a sub-collection of a Definite collection is itself Definite—then Separation (Ausserordnung) and Replacement seem to follow from Indefinite V. If we grant that finite concepts are Definite, we get Null Set and Pairs. If we restrict quantifiers to pure sets, Foundation follows, and perhaps Choice is close to a truth of logic. Nevertheless, there is cause to doubt the suitability of Indefinite V for the neo-logicist purpose. For one thing, it is not clear that any purely logical characterization of indefinite extensibility can be given. As the notion has been explained here, indefinite extensibility proper stands opposed to up-to-λ extensibility and a characterization of the latter will naturally demand, over and above the resources of higher-order logic, the ideology of the theory of the ordinals (which would activate an objection along the lines of Clark (2000), discussed at the close of Section 10.8 above). In this respect, then, Indefinite V, at least on the characterization of indefinite extensibility offered here, marks a step back from New V. (We shall return to this question in Section 10.13.)

Even if that problem can somehow be surmounted, however, the most serious foreseeable difficulty remains, as in the case of New V, with comprehension and, in the first instance, with Infinity. New V had the problem that, unless it is somehow independently given that the universe is more than countable, any infinite concept may consistently be taken to be bad and so not at the service of the abstraction of a well-behaved infinite set. Indefinite V looks certain to be encumbered with an exactly analogous problem: unless it is somehow independently given that some infinite concept is Definite, any infinite concept may consistently be taken to be bad, that is: indefinitely extensible, and so, again, not at the service of the abstraction of a well-behaved infinite set. But that is the issue about the prospects for whose resolution we just, in the previous section, concluded a pessimistic discussion. The idea that a strictly neo-logicist construction of a decently strong set-theory might proceed via second-order logic and Indefinite V comes, in effect, to the thought not merely that Dummett’s Aristotelianism might be refuted using just those materials but more, that any position could be similarly refuted which—from a classical standpoint—places indefinite extensibility anywhere short of the inaccessible.

There is a general problem of formulating a restriction of Law V which is both consistent and strong enough to develop a theory as strong as Zermelo–Fraenkel set theory (perhaps together with other abstraction principles). As with other attempts to develop a consistent (or even dialetheic) ‘naive’ set theory, that is non-trivial and sufficiently powerful, we have to add analogues of certain of the ZF axioms explicitly. In the case of Indefinite V, we seem to require an explicit axiom that there is a Definite infinite concept, to get Infinity. Similarly, we have to explicitly add an axiom that if a concept of sets is Definite, then so is its union and powerset.¹⁸ None of these follow from the general characterization of indefinite extensibility.

¹⁸ In a sense, this is the moral of Shapiro (2003a).
10.11 QUANTIFYING OVER INDEFINITE EXTENSIBLE TOTALITIES

We at last directly broach the topic of this volume. The question, simply, is whether it is ever appropriate or intelligible to speak of all of the items that fall under a given indefinitely extensible concept. Can we talk about all ordinals, or all cardinals, or all sets? The discussion for this and the next section will be organized through three closely related issues, focusing respectively on whether unrestricted quantification over the instances of an indefinitely extensible concept is intelligible, whether it is legitimate, and how—if it is to be both intelligible and legitimate—it requires to be understood.

There can hardly be any question about intelligibility from the extreme liberal point of view of Cantor and Zermelo (assuming that the viewpoint itself is intelligible, of course). If each particular transfinite cardinal, ordinal, and inaccessible rank, exists as an actual infinity, then they all do. Or so it would seem. The talk of the ‘potential’ infinity of the transfinite numbers (a la Cantor) or the inaccessible ranks (a la Zermelo) is just a picturesque way of saying that there is no set of all such numbers or ranks. But they do all actually exist, even if there is no set of them all and, it seems, we have just talked about them—all of them. Just in the very act of calling them indefinitely extensible, we somehow quantify over all of them, don’t we? What is the problem?

Extreme conservatism, it seems, must surely grant intelligibility too, albeit for quite different reasons. Dummett holds that the natural numbers and certainly the real numbers are indefinitely extensible. Someone who agrees with him but holds that we cannot legitimately have quantifiers ranging over any indefinitely extensible totality, would have to conclude that we can have no theory of arithmetic or analysis involving quantification over all the natural numbers, or all the reals—so no worthwhile arithmetic or analysis at all. That, of course, is not Dummett’s view. So he implicitly grants that one can intelligibly quantify over at least the natural numbers and the real numbers—so over at least those indefinitely extensible totalities. Dummett’s view has to be that arithmetic and analysis intelligibly and legitimately quantify over indefinitely extensible totalities.

The picture is more complicated, however. In a well-known passage in the last chapter of his (1991) Dummett suggests that Frege’s major ‘mistake’—what doomed him to Russell’s paradox—consisted in ‘his supposing there to be a totality containing the extension of every concept defined over it; more generally [the mistake] lay in his not having the glimmering of a suspicion of the existence of indefinitely extensible concepts’ (Dummett, 1991, 317). So now a reader might take it that Dummett thinks that there are at least some indefinitely extensible totalities over which one may not quantify. At least, she might draw that conclusion if she is also mindful of the many passages in Dummett’s writings in which he apparently endorses the idea that objectual quantification, if it is to be determinate in sense, requires the antecedent specification of a domain, i.e. a set of objects, over which the bound variables are to
range. If quantification of determinate sense requires antecedent specification of a domain; and if a domain is a set; and if indefinitely extensible concepts do not determine sets, then the reader will need no help to see what follows. But then what to make of the apparent concession that the quantifiers in arithmetic and analysis are in good standing? Is it that some legitimate quantification doesn’t require a domain? Or that some indefinitely extensible concepts do determine domains? Or what? We’ll return to this.

Boolos (1993, 222) takes it that the foregoing is Dummett’s line of argument: it would seem that [Dummett] does think that there has to be a — what to call it — totality? collection? domain? containing all of the things we take ourselves at any one time to be talking about. He would seem to believe that whenever there are some things under discussion, being talked about, or being quantified over, for example some or all of the ordinals, there is a set-like item, a ‘totality’, to which they all belong. That is, he supposes that whenever we quantify, we quantify not over all the (ordinals or) sets that exist but only over some of them . . .

I suspect that Dummett would agree . . . that whenever we use quantifiers, there must be some domain, some totality of objects, over which our variables of quantification range; so if we take ourselves to be quantifying over all classes, then we must assume that there is a totality or domain containing all classes. And it may be thought that it is part of what we mean by ‘quantify over’ that there must be some such domain. Certain textbooks may reinforce this impression by telling us that to specify an interpretation we must first specify a non-empty set (class, collection, totality), the universe of discourse (or domain) over which our variables range (p. 223)

In arguing for the legitimacy of unrestricted quantification, Richard Cartwright (1994, 7) dubs this presupposition the ‘All-in-One Principle’. It is of course a staple of contemporary model theory that each interpretation of a formal language contains a set to serve as a domain for the variables. In effect, Cartwright argues that this is only an artifact of the standard kind of model theory we use today, and not a regulative ideal for semantics generally. In present terms, the only conclusion to draw is that quantification over an indefinite extensible ‘totality’ is not covered by model theory. Boolos agrees:

If we look at the presentation of class theory found in Kelley’s General Topology (1955), we find that the theory presented there is a full-fledged theory of classes in which variables range over (pure) classes and in which ‘set’ is defined to mean ‘member of a class’ . . . Kelley’s axiom of extent (extensionality) reads, ‘For each x and y it is true that x = y if and only if for each z, z ∈ x when and only when z ∈ y.’ . . . Now, it seems to me that insofar as we have a grip at all on the use of the phrase ‘quantify over’, we have to say that Kelley, in laying down his axiom of extent, is quantifying over all classes (aggregates, collections). I take it that when

¹⁹ Dummett (1981, 567): ‘the one lesson of the set-theoretic paradoxes which seems quite certain is that we cannot interpret individual variables in Frege’s way, as ranging simultaneously over the totality of all objects which could meaningfully be referred to or quantified over. That is . . . why modern explanations of the semantics of first-order predicate calculus always require that a domain be specified for the individual variables . . . the one thing we may confidently say hardly any modern logician believes in is wholly unrestricted quantification. All modern logicians are agreed that, in order to specify an interpretation of any sentence or formula containing bound variables, it is necessary expressly to stipulate what the range of the variables is to be.’
Kelley says ‘each’, he means it. How else are we to understand the axiom of extent... except as saying that any classes \( x \) and \( y \) are identical iff \( x \) and \( y \) have the same members?

Why should we for a moment think that therefore there must be a collection of all the things that Kelley was using his variables to range over? If one checks the exposition of General Topology, at any rate, one will find no suggestion at all that there must exist some sort of super-class, containing all of the classes that the theory talks about... We can simply say: Our variables range over all classes

Or: over all (‘absolutely’, if you insist, all) objects there (‘really’) are. If Frege thought his variables could so range, as of course he did, he was not in error. (pp. 223–4)

Cantor’s and Zermelo’s texts also seem to presuppose that quantification over indefinitely extensible ‘totalities’: transfinite numbers or inaccessible ranks, is fully intelligible and fully legitimate. Consider, for example, the language in which Zermelo describes his program and in which his theorems are proved. What are we to make of his talk of ‘models’, ‘normal domains’ (i.e., inaccessible ranks), ‘order-types’, and the like? As noted, he proposed ‘the existence of an unbounded sequence of [inaccessible ranks] as a new axiom of “meta-set theory”.’ Again, the new principle states that for each ordinal \( \alpha \), there is a unique inaccessible cardinal \( \kappa_\alpha \). How are we supposed to interpret that except as talking about all ordinals, and all inaccessible ranks? Zermelo’s axiom is not meant as an assertion about some particular set-sized model but is surely intended to be taken at face value. The words in the meta-axiom are used, and not merely mentioned in a statement of satisfaction.²⁰

Characteristically, set-theorists are not content merely to quantify in the cases in point. Commonly, they introduce linguistic items that at least look like singular terms that stand for indefinitely extensible ‘totalities’. Cantor himself used ‘\( \Omega \)’ for the transfinite numbers; nowadays this symbol is used for (the von Neumann) ordinals. And ‘\( V \)’ is the accepted term for the pure iterative hierarchy. This much, to be sure, need not be particularly problematic. Typical uses of these literary devices are easily paraphrased away, in terms of predicates. For example, ‘\( \alpha \in \Omega \)’ is just shorthand for ‘\( \alpha \) is an ordinal’. The axiom V=L is just the statement that every set is constructible. One can reject the All-in-One principle, allow quantification over indefinitely extensible totalities, and still traffic in such apparent terms, provided they are only apparent—provided, as seems to be so, there is no pressing reason why expressions of the kind just noted should be conceived as standing for ‘Ones’.

Boolos and Cartwright seem to be insouciant about rejecting the All-in-One principle. However, as noted in Shapiro (2003), things are far from comfortable for a free-wheeling acceptance of indefinitely extensible quantification. Above, we defined an

²⁰ Of course, we can profitably ask about the set-theoretic models of Zermelo’s meta-axiom. A standard model of the meta-axiom would be a rank \( V_\lambda \) for which \( \lambda \) is a fixed-point in the series of inaccessibles: \( \lambda = \kappa_\lambda \). These fixed points are next (after ‘inaccessible’) in the series of so-called ‘small large cardinals’. Zermelo’s meta-axiom does not entail the existence of such an inaccessible. The natural next maneuver would be to postulate another axiom asserting the existence of an unbounded sequence of such fixed points. This would be a meta-meta-axiom, stating that the fixed points in question are themselves indefinitely extensible. Then we can inquire into the models of this axiom. The interplay between using principles like this and then studying their models (mentioning the principles) is rich indeed (see Drake, 1974).
'ordinal' to be the order-type of a well-ordering. The problem is that the very definition of a well-ordering seems, like the brooms of the Sorcerer’s Apprentice, to give rise to ever more well-orderings. It is, of course, routine to show that the relation of ‘less than’ on ordinals (or membership on the von Neumann ordinals) is itself a well-ordering: any sub-totality has a least element. But, of course, there is no order-type of the ordinals—no All-in-One in this case—on pain of contradiction. Yet is easy to define a two-place predicate that apparently characterizes a well-ordering that is strictly longer than $\Omega$: Let $\alpha$ and $\beta$ be ordinals. Say that $\alpha \prec_1 \beta$ if $\alpha \neq 0$ and either $\alpha < \beta$ or $\beta = 0$. That is, we make the order longer just by putting $0$ at the ‘end’. A routine trick. And why stop there? We can also define a relation that intuitively characterizes a well-ordering twice as long as $\Omega$: $\alpha \prec_2 \beta$ if either $\alpha$ is a limit ordinal and $\beta$ is a successor ordinal, or both $\alpha$ and $\beta$ are both limits and $\alpha < \beta$, or both successors and $\alpha < \beta$. In $\prec_2$, the limit ordinals come before the successors, and the limit ordinals and the successor ordinals are each isomorphic to the ordinals (and are thus each indefinitely extensible), according to ZFC anyway. Finally, here is a well-order that is $\Omega$ times as long as $\Omega$ (if you can pardon the expression): let $<x, y>$ be the ordered pair of $x$ and $y$. If $\alpha, \beta, \gamma, \delta$ are ordinals, then let $<\alpha, \beta> \prec_3 <\gamma, \delta>$ if either $\alpha < \gamma$ or both $\alpha = \gamma$ and $\beta < \delta$.

Notice that the constructions here are somewhat independent of how ‘many’ ordinals one thinks there are. If one goes for a strict Aristotelian account, and maintains that all Definite totalities are finite, then $\Omega$, the property, totality, or whatever, of all ordinals will be what the set-theorist calls ‘$\omega$’. The above predicate characterizing $\prec_1$ would thus define $\omega + 1$, which, for the strict Aristotelian, is longer than the ordinals. One can go on to define $2\omega$, $\omega^\omega$, etc. (although the use of limit ordinals will not be available). Similarly, if someone allows the existence of all but only classically countable ordinals, then $\Omega$ will be what the classicist calls ‘$\omega_1$’, the first uncountable ordinal, and it will then be routine to write predicates that characterize well-orderings corresponding to $2\omega_1$, $\omega_1^2$, etc. In short, whether one is a staunch conservative like Leibniz or Dummett, an ultra liberal like Cantor and Zermelo, or something in between, one will still have his special $\Omega$, the property of being an ordinal. This ‘totality’—the ordinals themselves—will be well ordered, and one can seemingly define well-orderings longer than that.

On the surface, it is legitimate to do transfinite recursions and inductions over ordinals and, presumably, only over ordinals. Nevertheless, set theorists occasionally seem to invoke transfinite recursions and inductions whose ‘length’ is at least that of $\prec_2$, i.e., twice as long as the well-ordering of all ordinals. For example, the concept $L$ of being a constructible set is defined by transfinite recursion over all ordinals (i.e., of length $\Omega$). But set theorists go on to do transfinite recursions on $L$, which are also of length $\Omega$. So, in effect, we have a transfinite recursion of length $2\Omega$. But over what objects?

The following appears in a survey article on mouse theory:²¹

²¹ Thanks to Tim Bays for drawing our attention to some of the relevant literature here. Mouse theory is a very technical branch of abstract set theory. As just noted, ‘$L$’ is the constructible hierarchy, used in Gödel’s proof of the consistency of the axiom of choice and the generalized
We begin by constructing $L$ level by level. The first $\omega$ levels are exactly the hereditarily finite sets, the next $\omega^2$ levels are exactly the sets that are hereditarily countable in $L$, and so on. Now we ask ourselves what comes next.

(Schimmerling, 2001, 486–7)

Of course, this talk of ‘construction’ is, as usual, only a metaphor. What is literally true is that we define the constructible sets ($L$) by transfinite recursion over all ordinals. And proofs about $L$ invoke transfinite induction over all ordinals. This much is captured in ordinary mathematical language with unrestricted quantification over ordinals. But what is the literal meaning of Schimmerling’s question, ‘what comes next’? What can possibly come after the ordinals? He continues:

For although we have climbed up to the minimal transitive proper class model of ZFC, foundational considerations that fall under the category of large cardinals have tempted us to adopt certain theories that extend ZFC. These extensions are not true in $L$, for they imply that there exists a non-trivial elementary embedding $j:L \to L$, which is known to fail in $L$. How do we continue or revise the construction in a way that buys us the existence of such an embedding? One naïve idea is to continue the construction past all the ordinals and throw in the proper class $j$ at stage $\omega_1'$ or beyond, but this approach leads to some obvious metamathematical problems that we find annoying.

What was that? — We are to go past all of the ordinals? Metaphor or not, one fears a lapse into nonsense. If there is a ‘past all the ordinals’ to ‘go on to’, we have not gone through all of the ordinals—through all of the well-ordering types. With ‘constructions’ like these what we surely have gone beyond are the ‘limits of thought’!

Typically, the way around the ‘annoying’ meta-mathematical problems to which Schimmerling refers is to replace the long transfinite recursions with codings. That is, the set theorist works hard to simulate the long transfinite recursion within ordinary, first-order set theory. Nevertheless, it seems to us that this grand transfinite recursion is coherent as it stands, or at least as coherent as anything else in set theory. If we can indeed legitimately and intelligibly talk about all ordinals, then, as we saw above, it is straightforward to define a predicate that characterizes a relation of order-type $2\omega$. The pairs of ordinals that satisfy this predicate also satisfy the second-order predicate of being a well-ordering. So why can’t we do transfinite recursions and inductions over them? Using variables or schematic letters, we can even do a transfinite recursion along the order-type $\prec_3$ above, of length $\omega_2$. And of course, this is not as far as we can go. There is a predicate corresponding to the ‘order-type’ of any polynomial involving $\omega$, essentially reproducing what Cantor proposed with $\omega; \omega_2 \omega_1'$ is not out of reach.

Of course, one must be careful how things are put, to avoid the obvious contradiction. If we say that the recursion has length $2\omega$, as in our informal gloss, then we

continuum hypothesis. The notion is defined by transfinite recursion over ordinals: $L_0$ is the empty set; for each ordinal $\alpha$, $L_{\alpha+1}$ is the set consisting of $L_\alpha$ together with all sets definable in terms of $L_\alpha$. If $\lambda$ is a limit ordinal, then $L_\lambda$ is the union of all $L_\beta$, with $\beta < \lambda$. A set $x$ is constructible if there is an ordinal $\alpha$ such that $x \in L_\alpha$. See any text for set theory, such as the excellent Jech (2002), for details.
are saying that there is a ‘past all the ordinals’. As noted, one can define a two-place predicate in the language of (first-order) set theory whose extent is the putative well-ordering in question (so to speak). The recursion and induction is done over that predicate, or over the ordinals (or pairs of ordinals) that satisfy the predicate. The ‘pain’ of contradiction comes, of course, if we think of this predicate as defining an order-type, a ‘length’, or any other sort of all-in-one. So the talk of the ‘length’ of the procedure, \((2\Omega, \Omega^2, \text{etc.})\), is only a metaphor.\(^{22}\) The legitimacy of the technique is a working hypothesis.

At this point, one might protest, paraphrasing Boolos (1998a, 35):

Wait a minute! I thought that set theory was supposed to include a theory about all, ‘absolutely’ all, the well-orderings and transfinite recursions that there are and that ‘well-ordering-type’ was synonymous with (or at least coextensive with or isomorphic to) ‘ordinal’.

Well, indeed. But the problem, to stress, is that predicates corresponding to these ‘order-types’ are definable as soon as we make the assumption that we can talk about—have bound variables ranging over—all ordinals.

Once again, the defender of absolutely unrestricted quantification (the ‘absolutist’) can—and presumably must—claim that there are no ‘objects’—no ordinals—that correspond to these explicitly definable long well-ordering predicates. Just as indefinitely extensible concepts determine no ‘Ones’, so these predicates simply have no associated order-types. The grounds for the Boolosian rejection of proper classes, endorsed earlier, must also preclude order-types \(2\Omega, \Omega^2, \Omega^2, \text{and the like, provided that these are construed as objects.}\)

Shapiro (2003) tentatively proposes what is, admittedly, a thin straw for the absolutist to grasp. The key observation is that the definition of a property (or predicate) being a well-order is second-order (see Shapiro, 1991, §5.1.3). So the absolutist can avoid the issue by demurring from using second-order variables in theories whose first-order variables range over an indefinitely extensible totality. Then the notion of a ‘class-sized well-ordering’ cannot even be formulated—and there will be no formula that expresses the seemingly patent facts that the ordinals are well-ordered, and that the formula that expresses \(2\Omega\) defines a well-ordering. Shapiro’s proposal would block the construction of mice in the unrestricted theory of the entire range of sets, ordinals, and models (at least if the text is taken literally).

\(^{22}\) An advocate of the Zermelo program can, of course, think of the definition of mice as restricted to (or as ‘taking place within’) a fixed arbitrary model \(M\) of set theory. The ‘totality’ of von Neumann ordinals in \(M\) is, of course, not a member of \(M\), but the ordinals in \(M\) do constitute a set, and thus a von Neumann ordinal, in all later models in the hierarchy (thanks to the ‘meta-axiom’ above asserting that the models of set theory are themselves indefinitely extensible). In effect, we rely on later models in the series to sanction the long transfinite recursions in \(M\) (see note 15 above). We saw above, however, that Zermelo’s own text has variables that range over all models of set theory, and thus, all ordinals whatsoever. That text is used to describe the hierarchy of models and to prove things about it. We suggest, at least tentatively, that it is legitimate to develop mouse theory on the entire hierarchy. One can write down a formula representing the definition of mice-in-the-hierarchy, and we can do the transfinite induction needed to show that it works.
It is a thin straw, though. Even putting to one side the (in our view) compelling arguments in favor of second-order languages in Shapiro (1991), Boolos’s later writings (e.g., 1984, 1985, 1985a), and elsewhere, the general point remains that denying the existence of the long well-orderings $\Omega$, $2\Omega$, $\Omega^2$ (as objects) merely seems like an ad hoc maneuver. As noted, one can define the long well-orderings (if that is what they are) as soon as the notion of ‘ordinal’ has been defined.

Boolos himself (e.g., 1984, 1985, 1985a) provides another way out. We can think of the formulas defining the long well-orderings as pluralities. There is no ‘thing’ that corresponds to, say, $2\Omega$, but we can talk about the ordinals, in the plural, that satisfy the relevant defining formula. We do the transfinite recursion over them (see also Agustín Rayo’s contribution to this volume).

But what if we just let them be? What, exactly, is wrong with the long transfinite recursions and inductions? Why can’t we just introduce such ‘ordinals’, or names for them, by suitably expanding our ontology? We just introduce a singular term, like ‘$\Omega$’, that is to denote the order-type of all ordinals, without intending to paraphrase it away. This gives rise to another singular term for $2\Omega$, another for $\Omega^2$, and one for $\Omega^\Omega$, and off we go. In doing so, we are just giving genuine names to well-orderings that we are capable of understanding and using, and treating those well-orderings as objects. The resulting theory is consistent if standard set theory together with an axiom asserting the existence of an inaccessible cardinal is. The envisioned theory is thus coherent, and, moreover, it seems to be true when $\Omega$ is interpreted as the order-type of the ordinals of ordinary set theory. So what’s wrong?

Well, again, simply that we have contradicted the understanding of $\Omega$ with which the process starts. $\Omega$ is the series of all ordinals—all possible order-types. Not all the ordinals except those that come after, the ‘proper’ ordinals, or higher-ordinals, or whatever. All of them. There may be a consistent formal theory, but it does not sustain the intended interpretation except at the cost of informal paradox—the same old Burali–Forti paradox—and for good measure, some additional variations on the theme—that was there all along.

Let’s stop running in circles and step back. So far as we can see, there are exactly five possible positions (at least, for anyone inclined to accept ordinals as objects; nominalism is another response, but cannot be considered in proper detail here). They are as follows:

In his contribution to this volume, Geoffrey Hellman argues that considerations like those broached here point toward nominalism. For Hellman, set theory is understood in terms of what collections, and what well-orderings, are logically possible. For formal details, see Hellman (1989) (and Parsons, 1977). For the usual reasons, there is no largest possible well-ordering: any possible well-ordering can be extended. By refusing to reify possible collections and possible well-orderings, the nominalist avoids temptation, or is at least less tempted, to postulate a (possible) order-type of all possible well-orderings. The situation with constructions like that of mice shows that we can study the properties of a given well-ordering by considering even longer well-orderings. No problem with that. It is analogous to limiting the construction to a single model in Zermelo’s hierarchy. In Hellman’s system, the definition of long well-orderings—over all possible models—is blocked by a formation rule that does not allow free second-order variables to occur within the scope of a modal operator. This restriction is the analogue of the above proposal, from Shapiro (2003), of not allowing bound higher-order variables when the first-order variables range over an indefinitely extensible
(i) Reject the intelligibility/legitimacy of quantification over all ordinals. In this case, the troublesome predicates, like ‘≺₁’, ‘≺₂’, ‘≺₃’, cannot be defined, and so the issue of whether they have order-types does not get off the ground. **Cost:** we cannot express what seem to be not only perfectly intelligible but *true* thoughts about the ordinals in general. Indeed, presumably we cannot even use plural expressions like ‘the ordinals’ (for if we could *refer* to them—to them all!—what could possibly prevent legitimate quantification?)

(ii) Allow the intelligibility/legitimacy of quantification over all ordinals but deny oneself second-order resources (Shapiro, 2003). The predicates ‘≺₁’, ‘≺₂’, ‘≺₃’, can be defined, but we cannot state, much less prove, that they are well-orderings. **Cost:** abrogation of what are arguably perfectly sound and legitimate expressive resources.

(iii) Allow the unrestricted quantifications and the definitions of the troublesome predicates, but deny that they are associated with ordinals (order-types). **Cost:** transfinite inductions and recursions of the relevant ‘lengths’ then come into question (at least on the assumption that transfinite recursions and inductions require an associated order-type) which are part of expert practice and seemingly quite intelligible. Perhaps more importantly, the resulting stance is open to the objection, stressed in Section 10.9, that it amounts to an unprincipled/casuistic restriction on principles (that every kind of well-ordered series has an order-type, that every initial segment of the ordinals has a limit) without which we don’t get the ordinals, liberally conceived, to fly the first place.

(iv) Allow the quantification and the predicates, allow the associated order-types, but deny that they are ordinals as originally understood—rather, they are ‘higher-order’ ordinals, ‘proper’ ordinals, ‘super-ordinals’, or whatever. **Cost:** Hypocrisy. Recall that Ω was supposed to encompass the ordinals in a *maximally general* sense of ordinal, common to all types of well-orderings. Also, the option is unstable. If we are now saying that Ω does not encompass a maximally general sense of ordinal, and that we need to distinguish (how many?) successive orders of ordinals, then just consider all of these, and the dialectical situation repeats itself, only without this fourth option.

(v) Allow the quantification and the predicates, allow the associated order-types, allow that they are ordinals as originally understood, . . . and just accept that there are ordinals that come later than all the ordinals. **Cost:** none—unless one demurs from the acceptance of contradiction.

Frankly, we do not see a satisfying position here. It seems that every one of the available theoretical options has difficulties which would be justly treated as decisive against it, were it not that the others fare no better. Such situations are not unprecedented in philosophy, but this one seems particularly opaque and frustrating. Since it is impossible to advance any particular response with any degree of conviction, any unqualified profession that unrestricted quantification—or quantification over ‘totality’. Perhaps the maneuver is less ad hoc here, since Hellman’s restriction is independently motivated.
indefinitely extensible totalities—is perfectly intelligible and legitimate seems to us misplaced. Unrestricted quantification is one component in the aporetic situation latterly reviewed. Like the others, it merits watchfulness.

10.12 DUMMETT’S ‘NEW ARGUMENT’ AGAINST CLASSICAL QUANTIFICATION

We return to the question of how Dummett’s attitude to indefinitely extensible quantification is to be interpreted. As noted, it cannot plausibly be that quantification over indefinitely extensible totalities is simply impermissible, or unintelligible since, as we have several times had cause to observe, Dummett regards even the natural numbers as indefinitely extensible and is nevertheless quite content to endorse intuitionistic arithmetic (which of course treats exactly the same class of sentences as meaningful as classical arithmetic does). In fact, however, there is a reasonably clear position to be extracted from Dummett’s remarks (and which is pretty much explicit in his (1994)). The view can be summarized in two claims: first, the Aristotelian claim that infinity is always merely potential (though perhaps with some modest degree of relaxation to allow at least some Definite countably infinite totalities); second, the claim that where quantification over an indefinitely extensible totality takes place, it cannot legitimately be understood classically. The relevant aspect of the classical understanding of the quantifiers is that they are in effect conceived as truth-functions, logical product and sum, issuing in statements which are determinate in truth value whenever all their instances are. In Dummett’s view, the appropriateness of such a conception of the meaning of quantified statements lapses as soon as their range becomes indefinitely extensible. In such cases, we do better to work with the broad model of the content of quantified statements proposed by the intuitionists—in effect, the inferentialist model pioneered in the work of Gentzen and refined by Prawitz—against a background in which the principle of bivalence is dropped.

Here we shall have nothing more to say about the generally intuitionistic direction which Dummett proposes. Our concern is purely with the first step towards it, the claim that classical quantification misconstrues the legitimate content of quantification where indefinitely extensible totalities are concerned. The matter has received much discussion, and requires more, but we shall here attempt no more than to exclude a natural line of misinterpretation and to offer a suggestion about how, we believe, Dummett’s position may be better understood.

Consider the following set-up. We construct a long strip of paper—perhaps as much as twenty meters in length—whose color starts out scarlet at one end but then fades very gradually and seemingly continuously to a yellowish-orange at the other. It is a Sorites strip, if you will. On it are inscribed a series of randomly selected decimal numerals, in Times 10 point font, as close together as they can be consistently with their ready distinguishability one from the next. Consider the statement (A) ‘All the numerals on a red background denote multiples of 7.’ Each instance of (A) is decidable and determinate in truth-value. But that is, plausibly, totally insufficient for the conclusion that (A) itself has to be determinate in truth-value. It is insufficient for
the obvious reason that it is not fully determinate which the instances are. The phrase ‘numeral on a red background’ is vague.

That this, or something like it, might approximate Dummett’s thought is encouraged by his apparent adherence to the All-in-One principle, at least for quantification as classically understood. To require, it might be suggested, that quantification have the back-drop of a specified domain—a specified set of objects to constitute the range of the quantifiers—is tantamount to the requirement that it be Definite to what population of objects the quantified statements in question are to be accountable. If no domain is specified, we run the risk of indeterminacy in the range of admissible witnesses, and thereby indeterminacy in the truth-conditions of what we say. This is what happens in the case of the Sorites strip. ‘Numeral on a red background’ exactly fails to specify any determinate set of inscribed numerals, with indeterminacy in the truth-conditions of quantifications over them the immediate result.

Dummett does, it is true, sometimes speak of indefinite extensibility as a kind of vagueness (e.g., Dummett, 1963, *passim*). But the foregoing rather simple-minded proposal had better not be the intended argument. What is vague in the Sorites strip is where the numerals-on-red stop and the others begin. The quantified statement (A) is vague because there is no sharp cut-off between the numerals that satisfy its antecedent and those which do not. But nothing like that is true of the ordinals, to stay with our paradigm, however liberal or conservative one may be—not unless we really do think we can attach a sense to the idea of ‘going past’ all the ordinals. Even then the analogy limps, since there will be a determinate first element of whatever other sort of thing we are pleased to postulate—a first ‘proper’ ordinal, or whatever. The key component in the analogy of the Sorites strip is that the numerals on red are indeterminate in extent within a wider population of numerals on the strip. No counterpart of that features in anyone’s conception of the ordinals (even someone tempted by the prospect of ‘proper’ ordinals)—and in particular not in Dummett’s.

Is that feature, though—indeterminacy of extent within a wider population—essential to the intended point? Dummett (1991, p. 316–17) writes:

> Better than describing the intuitive concept of ordinal number as having a hazy extension is to describe it as having an increasing sequence of extensions: what is hazy is the length of the sequence, which vanishes in the indiscernible distance.

It is true that it is indeterminate how far the numerals-on-red extend within the Sorites strip of numerals, and that that is not the way to describe the indeterminacy of the extent of the ordinals—there is no ‘larger strip’ on which they peter out. But still, it may be suggested, they are indeterminate in extent. Hence the stand-off between conservative and liberal positions. It is conceptually open whether to regard them as confined to the finite, or the recursive, or the countable, or the accessible, or . . ., or whether we let them rip, stopping only when the apparatus buckles on Burali–Forti. And if this matter is conceptually open, then there is still going to be potential indeterminacy in the truth-conditions of quantifications over all ordinals, when conceived classically as functions from aggregates of truth-values to truth-values. There will be such indeterminacy because it is indeterminate what goes into the argument pool. All we can legitimately say is that such statements may be counted true provided it
is guaranteed they will hold *no matter how far the series is legitimately taken to extend*; and false if exceptions are guaranteed under the same hypothesis.

Yet this interpretation too—however much his own remarks may encourage it—cannot be true to Dummett’s intent. It cannot be true to his intent because he himself has a *position* on the issue in question—the position of Aristotelianism (or something close to it). The train of thought outlined is at best impressive for a theorist who lacks a position, who views the extent of the ordinals as open. A theorist who, for good reasons or bad, takes the view that Cantor’s second principle fails for the totality of ordinals of such-and-such a kind, has no motive to sympathize with the argument, even though he may allow that the totality in question is indefinitely extensible. So the connection we seek has still to be made out.

We do not suggest that Dummett is confused on the matter. But we do suggest that the comparison between indefinite extensibility and forms of vagueness, or indeterminacy, is badly conceived. More accurately, the comparison is misleading when taken as an invitation to think about the alleged counter-classical implications of indefinite extensibility on the model of (what would be widely accepted as) the counter-classical implications of (one or another form of) vagueness. The right comparison is only in the *effect*: both indefinite extensibility and vagueness may be held to call into question the validity of the principle of bivalence for a relevant range of statements. So the question still remains: *why* (might it be supposed that) indefinite extensibility has that effect?

We have no answer to offer which effectively clarifies whether Dummett is right or wrong, but we think we know where to look. The key to understanding his argument has to be found in the implicit comparison between the operation of set-formation and the operation of universal quantification (for example) as classically conceived. Think of a set as the value of a many-one function that takes exactly the elements of the set as argument. If the objects of some kind are indefinitely extensible, the set-function cannot generate a set of them all—for any value it can give will immediately be at the service of the definition of a new object of the appropriate kind, demonstrably excluded from the set in question. Now think of quantification in similar terms, as a many-one function that yields a truth-value when given a range of instances as argument. If the elements which the instances concern are indefinitely extensible, then no application of the function can embrace them all—for any collection of the instances to which it is applied will immediately be at the service of the definition of a new instance, so far unreckoned with.

The crucial thought is thus that a function requires a *stable* range of arguments if it is to take a determinate value. Any vagueness, then, in the extent of an indefinitely extensible concept is not really the point. The operation of classical quantification on indefinitely extensible totalities is frustrated not because it is vague what the arguments are, but because any attempt to specify them subserves the construction of a new case, potentially generating a new value. The reason why quantification, classically conceived, requires a domain—a Definite totality—to operate over is just that.

As stated, our purpose here is only to try to locate the real issue at stake in Dummett’s ‘new argument’, not to take sides. It certainly is robustly part of the classical, model-theoretic semantics of quantification to see it as a function,
standardly set-theoretically conceived, whose argument- and value-ranges are accordingly likewise sets. Since indefinitely extensible concepts do not determine sets, that much of classical semantics is certainly in jeopardy in the present context (as noted already, in Section 10.11 above). But the question whether quantification over the instances of such concepts may legitimately be viewed nonetheless as determinate is still open. Whether the last line of argument proposed can be developed strongly enough to force the issue is left for another occasion.

10.13 INDEFINITE EXTENSIBILITY AND REFLECTION

The passage quoted from Tait (2000, 284) at the end of Section 10.9 above captures a widely held conviction that the iterative hierarchy in its entirety is ineffable. Any attempt to characterize it uniquely not only fails, but provides us with the resources to characterize more sets. This suggests a reflection principle: for any formula $\Psi$ in the language of set theory, if $\Psi$ is true (in V, so to speak) then there is a set that satisfies $\Psi$ — anything true of all the sets is true of the elements of some set. If the variable $x$ does not occur in $\Psi$, then let $\Psi^{[x]}$ be the result of restricting the first-order variables in $\Psi$ to $x$, restricting the monadic second-order variables to subsets of $x$, etc. If $\Psi$ does not contain the variable $x$, then the following is an instance of the reflection principle:

$$\Psi \rightarrow \exists x \Psi^{[x]}.$$ 

Each instance of the reflection principle in which $\Psi$ is first-order is already a theorem of ZFC. Instances of the reflection principle in which $\Psi$ is higher-order entail the existence of so-called small large cardinals. For example, we note that the conjunction of the axioms of second-order set theory are true, and so, by reflection, there is a set that satisfies these axioms. The set or ordinals contained in such a set is a strongly inaccessible cardinal. Thus, the principle entails the existence of an inaccessible cardinal (see Lévy (1960 and 1960a); Shapiro (1987 and 1991, Chapter 6)).

So the reflection principle is a substantial set-theoretic thesis. Bernays (1961) shows how many of the axioms of set theory, plus some small large cardinal principles, can themselves be deduced from a strong version of the reflection principle, and hence that it can play a unifying role in axiomatic set theory (see Burgess, 2004).

Now, the reflection principle has impressed a number of theorists as of-a-piece with, or at least implicated in, the indefinite extensibility of the iterative hierarchy, at least intuitively. This suggests a different way in which indefinite extensibility might

Note that this argument presupposes the legitimacy of second-order statements as applied to the entire iterative hierarchy (V), contrary to the tentative proposal in Shapiro (2003) noted above. We have also had occasion to note that ordinary model theory does not allow for interpretations in which the bound variables range over an indefinitely extensible ‘totality’. Shapiro (1987) shows that with a reflection principle, the restriction does not change the extension of validity. The idea is that if an indefinitely extensible ‘totality’ shows an argument to be invalid (i.e., true premises, false conclusion), then, by reflection, there is a set that also shows that argument to be invalid. So, in a sense, a reflection principle is a presupposition of the use of set-theoretic model theory as the semantics of higher-order languages.
have a bearing on the construction of foundations for a strong set-theory. Rather than seek an appropriate abstraction principle, or principles, to reflect the indefinite extensibility of the sets, in the fashion under scrutiny in Section 10.10 above, perhaps one can appeal to indefinite extensibility to ground the reflection principle as a special axiom to be used alongside whatever other foundational principles—they can as well be abstraction principles—are motivated by one’s theoretical standpoint. This is a strategy explored by John Burgess in his (2005).

However the connection between reflection and indefinite extensibility seems to us fugitive on closer scrutiny. Burgess (2004, 2005) proposes the following heuristic train of inter-connections leading from one to the other: ²⁵

(1) . . . the sets are indeterminately or indefinitely many. (Indefinite Extensibility)
(2) . . . the sets are indefinably or indescribably many.
(3) . . . any statement \( \Phi \) that holds of them fails to describe how many they are.
(4) . . . any statement \( \Phi \) that holds of them continues to hold if reinterpreted to be not about all of them but just about some of them, fewer than all of them.
(5) . . . any statement \( \Phi \) that holds of them continues to hold if reinterpreted to be not about all of them but just some of them, few enough to form a set. (Reflection)

Burgess is, of course, fully aware that these transitions are not immune to challenge. It seems to us that the move from (1) to (2) in particular needs more motivation. The idea that the pure sets or the ordinals are indefinitely extensible does not entail, or even suggest, that they cannot be described distinctively. As above, indefinite extensibility only says that any Definite collection of sets or ordinals can be extended. To make the connection from (1) to (2) one might try to argue for the contrapositive: that if a given sentence is true of the \( B \)'s, and of no proper sub-collection of them, then the \( B \)'s are Definite. But, right now, we do not see how to construct a plausible such argument.

The statement (3) seems problematic. As noted above, the very notion of indefinite extensibility, and the statement that the sets are indefinitely extensible, is itself neutral on the matter of how conservative or liberal one is on the question of ‘how many’ ordinals (etc.) there are, so to speak. It is not part of the meaning of ‘ordinal’, nor is it part of indefinite extensibility as such that one cannot describe the totality of the ordinals. A staunch Aristotelian or one who thinks that all sets are accessible can hold that the sets are indefinitely extensible, and have no trouble describing ‘how many’ of them there are. The former gives a second-order formula that is satisfied by all and only countably infinite properties; the latter uses a formula that holds of all and only properties the size of the first inaccessible (see Shapiro, 1991, §5.1.2).

It seems to us that the putative connection with reflection is best made under the aegis of the ultra-liberal view of Cantor, Zermelo, and most practicing set theorists, namely, the view that only inconsistency will be allowed to keep us from going on. In

²⁵ The reader should be aware that Burgess goes on to state that he, ‘like Boolos, [has] no use for Michael Dummett’s notion of indefinite extensibility’. The sketch he offers is for those who do have some use for the notion.
keeping with the above passage from Tait (2000, 284), our liberal might claim that any non-trivial criterion \( C \) for, say, being an ordinal, would not show that all and only ordinals have \( C \). Instead, she takes the criterion to show that there are ordinals that lack \( C \). For example, an opponent might suggest, with the staunch conservative, that being finite is a criterion for being an ordinal. No, says the ultra-liberal. This shows that there are ordinals that are not finite. The semi-conservative suggests that being accessible is a criterion for being an ordinal. After all, every ordinal we have defined so far is accessible. No, says the liberal. This shows that there are ordinals with inaccessibly many predecessors.

It is straightforward to interpret the liberal as appealing to reflection at each attempt of a conservative opponent to corral the ordinals and the sets. When the staunch conservative claims that being finite is a criterion for being an ordinal, the liberal notes that there are infinitely many (finite) ordinals. So, by reflection, there exists an infinite set and thus an infinite ordinal. The semi-conservative concedes this, but then claims that every ordinal is accessible. The liberal notes that there are inaccessibly many ordinals thus conceded. So, by reflection, there is an inaccessible set and thus an ordinal with inaccessibly many predecessors.

The reflection principle itself has an extensibility that is of a piece with the liberal perspective. We noted just above that the reflection principle entails the existence of an inaccessible cardinal. We apply the reflection principle to the statement that there is an inaccessible cardinal. This gives us the existence of a standard model \( m \) of set theory that itself satisfies this statement. The set of ordinals in \( m \) is an inaccessible that contains an inaccessible as a member. Thus we have the existence of two inaccessibles. Similar repeated applications of the reflection principle yield the existence of 2, 3, \ldots inaccessibles. Then, repeating in the transfinite (so to speak), we show that there is an \( \omega \)th inaccessible, a fixed point in the hierarchy of inaccessibles, and so on through small large cardinals (short of the so-called indescribable cardinals, which are the smallest models of the reflection principle, see Shapiro 1987).\(^{26}\)

As Tait says, one can peel off Cantor’s journey at any point. In a sense, the reflection principle expresses our intent to not do so unless consistency demands that we do. Any attempt to ‘peel off’ too early just gives rise to more, previously unheard of, ordinals. The principle represents another of Priest’s (2002) ‘limits of thought’: the limit of the expressible.

In any case, even if indefinite extensibility (with or without the liberal orientation) neither entails nor is entailed by reflection, there are interesting connections bearing on matters discussed earlier. One concerns the prospects for neo-logicist set-theory based on Indefinite V discussed in Section 10.10. Noting the role of limitlessness in the concept of indefinite extensibility, and the seemingly unavoidable need to invoke the ideology of the ordinals in characterizing that, we expressed scepticism about the logical definability of indefinite extensibility, prerequisite for the logicism in ‘neo-logicism’. But the reflection principle enforces a stronger pessimism: that one cannot characterize the notion of indefinite extensibility even given all the expressive

\(^{26}\) The play with Zermelo-type meta-axioms in note 20 above can also be seen as applications of reflection.
resources provided by set theory. For if we assume that each pure set is Definite and that the sets themselves are indefinitely extensible, then it follows from the reflection principle that there is no formula $\Phi(X)$ in the language of set theory in which all quantifiers of $\Phi$ are restricted to $X$, such that for each concept $P$, $\Phi(P)$ holds if and only if the $P$’s are indefinitely extensible. Let $\Psi$ be the result of instantiating $X$ with the universal concept in $\Phi$ (e.g., replace any subformula of the form $Pt$ with $t = t$). Then $\Psi$ is true of the sets. Thus, by the reflection principle, there is a set $v$ that satisfies $\Psi$. So $\Phi(P)$ holds when $P$ is the concept of being a member of $v$. But this concept is Definite.

The reflection principle also gives a surprising alternative resolution to the main matter treated in Section 10.8, concerning Hume’s Principle and ‘anti-zero’. Recall that in responding to one objection made in Boolos (1997), Wright (1999) accepted ‘the plausible principle . . . that there is a determinate number of $F$’s just provided that the $F$’s compose a set’, and so ‘there is no number of sets’, no number of cardinal numbers and no number of ordinals. Wright’s proposal was that the variables in HP be restricted to Definite (sortal) concepts.

But wait. It follows from the reflection principle that there is a global well-ordering of the set-theoretic hierarchy. For suppose that there is no such well-ordering. Then the second-order statement stating the existence of such an ordering (see Shapiro, 1991, ch. 5, §5.1.3) is false. So, by reflection, there is a set that lacks a well-ordering. This contradicts Zermelo’s well-ordering theorem. In sum, with the reflection principle, the usual axiom of local choice (the ‘C’ in ‘ZFC’) entails Global Choice, and (with Foundation) Global Choice entails the existence of a global well-ordering.

It follows from this that if $P$ is a concept of (pure) sets, then if there is no set that contains all of the $P$’s, then the $P$’s are equinumerous with the von Neumann ordinals. That is, in the presence of the reflection principle, any two indefinitely extensible ‘totalities’ of sets are equinumerous to each other.²⁷

So it follows from the foregoing principles that the unrestricted HP is satisfiable on the pure sets. Each set has an aleph as its cardinality, as normal, and there is in addition one (and only one) more cardinality for the indefinitely extensible totalities. Any non-aleph will do. Let’s call it $\infty$. So, after all, we can—correctly and intelligibly—say that there is a cardinal of all cardinals, and a cardinal of all ordinals, and a cardinal of all sets. It is just $\infty$. And if we can say this, then why shouldn’t we? Since all indefinitely extensible concepts of sets are equinumerous (as above), then why shouldn’t we associate them with a ‘size’, a size bigger than any set-size? If it is felt that ‘size’ should be restricted to sets, we can just use another name. We hereby introduce a new concept: the schmize of a set is the aleph that is equinumerous with it; and the schmize of an indefinitely extensible collection of sets is $\infty$. HP is then satisfied if we interpret ‘$Nx:Fx$’ as the schmize of the $F$’s.

²⁷ There is a similarity between the derivation of the existence of a global well-ordering here and that in Section 10.10 above, on neo-logicist set theory. Here the conclusion follows from reflection and the (local) axiom of choice, whereas the previous conclusion follows from New V alone, without local choice. Indeed, local choice also follows from New V, from global choice. So we have no immediate reason to think that the reflection principle runs afoul of Wright’s principles of conservation.
But what of the above claim, common to Wright and Boolos, that there is no cardinal of all cardinals? Well, one question, of course, is whether there is indeed compelling philosophical mandate for the reflection principle. Even if there is, there may still also be good philosophical reason—as Wright suggests—why the concept of cardinal number should be restricted to Definite concepts. But if so, the point can be accommodated by regarding HP as characteristic of cardinality only for the Definite case. So regarding it does not require that we limit the application of the N-operator to Definite concepts: we merely cease to regard it as connoting cardinality or ‘size’ when applied to the indefinitely extensible, secure in the knowledge that, given reflection, Hume’s Principle is satisfiable in any case.

Boolos’s charge was that HP is false on the pure sets. As we noted earlier, the way he presented the charge left some scope for maneuvering about its force. But Wright’s response was concessive, granting that there was philosophical reason to restrict the principle in any case. What we have just seen is that, given reflection, there is a response to the objection that avoids any such concession. Hume’s principle, to repeat, is satisfiable in the iterative hierarchy if reflection holds. The only philosophical issue is the proper interpretation of the N-operator. But we shall take matters no further here.

REFERENCES


Cantor, G. (1833) *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-

(1887) ‘Mitteilungen zur Lehre vom Transfinitten 1, II, Zeitschrift für Philosophie und
philosophische Kritik 91, 81–125, 252–70; 92, 250–65; reprinted in G. Cantor, *Gesam-
melte Abhandlungen mathematischen und philosophischen Inhalts*, (ed.) E. Zermelo, Berlin:


Abstraction, University of St Andrews.

Publishing Company.


Press.


(1994) ‘Chairman’s Address: Basic Law V’, *Proceedings of the Aristotelian Society* 94,
243–51.

Symbolic Logic* 27, 259–316.


Frege, G. (1884) *Die Grundlagen der Arithmetik*, Breslau, Koebner; *The Foundations of Arith-


Press.

Press.


pany.


Leibniz, G. (1863) *Mathematische Schriften von Gottfried Wilhelm Leibniz*, (ed.) C. I. Gerhart,

(1980) *Sammliche Schriften und Briefe: Philosophische Schriften*. Series 6, Vols. 1–3, Ber-

(1996) *New Essays on Human Understanding*, Ed. and Trans. by P. Remnant and J. Benn-
ett, New York: Cambridge University Press.

*Philosophical Review* 107, 49–96.

49, 1–10.

(1960a) ‘Axiom Schemata of Strong Infinity in Axiomatic Set Theory’, *Pacific Journal of

Parsons, C. (1977) ‘What is the Iterative Conception of Set?’, *Logic, Foundations of Math-
ematics and Computability Theory*, (ed.) R. Butts and J. Hintikka, Dordrecht: Holland,
D. Reidel, 335–67.


